

Research Article

Lithium-ion Batteries Solid-phase Diffusion Approximations, Transient Response, Electrolyte Concentrations delayed in Time Stability Analysis under Parameters Variation

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Abstract

Lithium-ion batteries are described by mathematical model which involve variables such as electrolyte concentrations which are volume averaged over respective region. Other possible dependent variables can be solution phase potential and solution phase concentration. The schematic Lithium-ion cell sandwich defines three important distances l_p , l_s , and l_n for positive electrode length, separator length, and negative electrode length, respectively. The perturbation techniques define the model and the number of solved equations. Lithium-ion batteries are characterized by three regions with different physical properties and characterized equations in each region. The Lithium-ion model characterized ordinary differential equations (ODEs) are converted to delay differential equations (DDEs) which reflect the electrolyte concentrations delayed in time due to inconsistency and interferences in the solid-phase diffusion approximations and transient response. The key model delayed variables are electrolyte concentrations which are volume averaged over respective region ($C_{\text{cathode}}^{\text{avg}}$, $C_{\text{sep}}^{\text{avg}}$, $C_{\text{anode}}^{\text{avg}}$). The three DDEs Lithium-ion model gives the prediction for electrolyte concentrations under influence of variables delay in time. The standard local stability analysis about any one of the equilibrium points of Lithium-ion batteries model is inspected for stability and stability switching behaviour under delay parameters in time variation.

Introduction

A lithium-ion battery is a unit which stores energy and is a rechargeable storage device. The Lit-ion device uses the movement of lithium ions between two electrodes, the positive and the negative. The Lit-ion batteries are very useful for consumer electronics and renewable grid storage. The Lit-ion batteries are known as high energy density devices. They have cathode and anode ports. The anode port is constructed from graphite which stores the lithium ions and lets them to be free during discharge process. The cathode is usually made of lithium metal oxides (NMC or LFP) and functions as a storage medium. Additional other two Lit-ion components are electrolyte and separator [1]. The electrolyte is a liquid or gel which carries the positively charged Lit-ions between the electrodes. The separator is a permeable physical barrier which keeps the electrodes apart. It prevents the short situation. By implementing charging process, the external electrical source forces the lithium ions to move back from the cathode to the anode. By implementing discharging process, the lithium ions move from the anode to the cathode. The movement is done through electrolyte by pushing electrons through an external circuit to power device. The

advantages by using Lit – ion batteries are the high energy density, very low self – discharge rates, and no “memory effect”. If we compare Lit – ion batteries to nickel cadmium (NiCd) batteries, there is possibility to partially charge Lit – ion devices at any time without losing usable capacity or altering their voltage. The reason of Lit – ion batteries “not remember” is based on the fact that there is no crystal formation and better partial charging ability (keeping Lit – ion battery to 20% - 80% capacity extends its lifespan). The Lit – ion disadvantages are high cost, lifespan of two to three years, temperature sensitivity which cause to poor functionality under extreme cold or overheating (extreme heat). The Lit – ion batteries additionally have high risk of fire (thermal runaway). It happened when they mechanically damaged or over charged. Extreme heat or overheating can cause a short circuit which is call “thermal runaway”. It is uncontrolled chemical reaction which creates intense fires or explosions. The Lit – ion batteries are more expensive compare to NiCd batteries and have limited lifespan. They are very sensitive to temperature changes which effect their functionality. The important in controlling Lit – ion batteries to get conclusion on state of charge (SoC) and state of health (SoH) statuses. The lithium – ion battery state of charge (SoC) is the level of available charge relative to its maximum capacity [2]. It is ranging from zero (empty state) to one hundred percent (full state). It is battery fuel gauge and related to battery chemistry characteristics. We measure Lit – ion battery SoC by using battery management system (BMS) which run dedicated algorithms. The SoC measurement methods are coulomb counting (integrating the battery flowing current in and out), open circuit voltage OCV (measure the resting voltage of the battery since the Lit – ion batteries have flat voltage curve for discharge cycle), and combination of two methods alongside with temperature sensors to get the highest precision and the desire safety. The Lit – ion battery state of health (SoH) measure the device current electrochemical performance compare to its original current. We mark battery SoH by percentage which gives information on battery’s maximum capacity decay and increasing internal resistance for battery’s lifetime. Three regions characterized Lithium – ion batteries, the first is positive electrode, the second is separator, and the third is the negative electrode. The model simulates Lithium – ion battery in real time. A spaced discretized equations describe the real time of Lit – ion batteries. Optimization process is needed for the model. If we have a battery uniform current distribution then Lit – ion electrolyte concentration is expressed by partial differential equations [3]. The main variable is the electrolyte concentration and other important parameters are pore wall flux for negative and positive battery electrodes (j_n and j_p , respectively). The battery pore wall flux is the rate which lithium ions move between the liquid electrolyte and the active solid material particles per unit area. It is boundary condition which related to coupling solid – phase diffusion and liquid phase electrolyte conservation equations. The source or sink of ions is determined by the pure well flux. It is implemented in both solid phases and liquid phases. The electrolyte concentration is the main Lit – ion variable when we want to inspect the battery operation dynamic. The electrolyte concentration variables are $C_{ave}^{cathode}$, C_{ave} , C_{ave}^{anode} and presented by ordinary differential equations (ODEs) in time. The flux (concentration derivatives) of Lit – ion battery two interfaces are located at $x = l_p$ and at $x = l_p + l_s$, respectively. The variable x is related to Lit – ion sandwich distance from the first location at $x = 0$ of positive electrode. We consider the battery concentration profile as a parabolic and the flux at the battery structure interfaces can be approximated. The Lit – ion battery solid phase diffusion approximations are reduced the computational efforts we need for real time BMS functionality. We get transient response of the model. The battery model helps us to predict the dynamical behaviour of the battery device in three regions which related to electrolyte concentration. The Lit – ion batteries discharge behaviour is the first outcome from battery model simulation. The Battery solid – phase diffusion coefficient (D_s) dives us indication how fast lithium ions are moving inside electrode particles. If we define in the process delay parameters ($\tau_i \forall i = 1, 2, 3$ for $\tau_i \in \mathbb{R}; \tau_i > 0$), they depend heavily on particle geometry, pulse duration, and relaxation period. Additionally, the Lit – ion electrode geometry modelling can define the internal time delay parameters which are diffusion time constants. It is for both microscopic and macroscopic scales. The defined model time delay parameters represent the actual physical time which required for Lit – ion battery lithium concentrations and electrical potentials to reorganize and get to steady – state (ss) equilibrium state during charge, discharge, or to the remain process periods. The Lit – ion battery electrolyte concentrations are delayed in time due to the solid – phase diffusion approximations and the transient response. The delayed in time parameters ($\tau_i \forall i = 1, 2, 3$ for $\tau_i \in \mathbb{R}; \tau_i > 0$) Influence the Lit – ion battery system stability and need to be inspected deeply by stability criteria and non – linear dynamic theory. The standard local stability and stability switching are influenced by system overall parameters and model delay parameters.

Lithium-ion batteries solid-phase diffusion Approximations, transient response, electrolyte concentrations delayed in Time

We can define the Li – ion batteries electrolyte concentration as volume averaged over specific region, where C is the electrolyte concentration, we get the specific region concentration as $C_{ave}^{cathode} = \frac{1}{L_p} \int_{x=0}^{L_p} C dx$, $C_{ave} = \frac{1}{L_s} \int_{L_p}^{L_p+L_s} C(x) dx$ and $C_{ave}^{anode} = \frac{1}{L_n} \int_{L_p+L_s}^{L_p+L_s+L_n} C(x) dx$, where $L_p = \frac{l_p}{L}$, $L_s = \frac{l_s}{L}$, $L_n = \frac{l_n}{L}$, and $L = l_p + l_s + l_n$. We have two interfaces in the Lit –

ion cell sandwich structure, first is at $x = l_p$ and the second at $x = l_p + l_s$. L is the overall length of the Lit – ion positive and negative electrodes plus the separator length. l_p is the length of positive electrode (thickness of the cathode), l_s is the length of separator (thickness of the separator), and l_n is the length of negative electrode (thickness of the anode). Typical values for l_p , l_s and l_n are 80×10^{-6} m, 25×10^{-6} m and 88×10^{-6} m, respectively [4]. The concentration derivatives (flux) are at two interfaces, $x = l_p$ and at $x = l_p + l_s$, respectively. We consider the Lit – ion concentration profile is parabolic and approximate the related fluxes. We can express the Lit – ion specific region concentrations by rate equations for getting the transient response of the Lit – ion model (Fig. 1). The related delay variables in time are $C_{ave}(t) \rightarrow C_{ave}(t - \tau_1)$, $C_{ave}^{cathode}(t) \rightarrow C_{ave}^{cathode}(t - \tau_2)$ and $C_{ave}^{anode}(t) \rightarrow C_{ave}^{anode}(t - \tau_3)$, $\tau_1, \tau_2, \tau_3 > 0$ and $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$.

The Lit – ion rate delay differential equations (DDEs) are $\frac{dC_{ave}}{dt} = \frac{-6\varepsilon_s^3(2C_{ave}(t - \tau_1) - C_1 - C_3)}{L_s^2}$; $\frac{dC_{ave}^{cathode}}{dt} = \frac{3\varepsilon_p^3(C_1 - C_{ave}^{cathode}(t - \tau_2))}{L_p^2} + J_p$ which related to positive electrode DDE for $C_{ave}^{cathode}$, and $\frac{dC_{ave}^{anode}}{dt} = \frac{3\varepsilon_n^3(C_3 - C_{ave}^{anode}(t - \tau_3))(1 - L_p - L_s)}{L_n^3} + J_n$ which related to negative electrode DDE for C_{ave}^{anode} . We consider that

the derivatives $\frac{dC_{ave}}{dt}$, $\frac{dC_{ave}^{cathode}}{dt}$ and $\frac{dC_{ave}^{anode}}{dt}$ do not affected by the delay parameters (τ_1, τ_2, τ_3). J_n and J_p are defined as the wall flux at the negative and positive electrodes, respectively [5]. In physics-based lithium – ion battery model ε_n and ε_p parameters are the volume fractions or porosities of the negative electrode (anode), and positive electrode (cathode), respectively. The negative Lit – ion battery ε_n parameter typical values are between 0.3 and 0.5 (30% to 50% porosity) values for negative electrode (ε_n). It is dependent on calendaring and whether it is a high – power or high energy cell. The positive Lit – ion battery ε_p parameter typical values are between 0.25 to 0.4 (25% to 40% porosity). We define X parameter as $X = \frac{x}{L}$, J_p parameter as $J_p = [L^2 \frac{(1 - t_+)}{c_0 D \varepsilon_p}] \times (\frac{I}{F l_p})$, J_n parameter as $J_n = [L^2 \frac{(1 - t_+)}{c_0 D \varepsilon_n}] \times (\frac{I}{F l_n})$. In Lithium – ion battery model, D parameter is the diffusion coefficient which is measured in $\frac{m^2}{sec}$ and represents how fast lithium ions move through

the electrode particles or the electrolyte. It dictates the battery's charge/discharge rate and power capacity. Typically, the D parameter value is 7.5×10^{-10} (separator). In Lithium – ion battery model, F parameter is Faraday's constant. It has typical value of $96487 \frac{C}{mol}$ (separator) and represents the magnitude of electric charge per one mol of electrons. Lithium – ion F constant is related to ionic transport, diffusion and the model reaction kinetics. In lithium – ion battery model, c_0 is the initial or reference concentration of lithium ions (measured by $\frac{mol}{m^3}$) [6]. The typical value of c_0 parameter is $1000 \frac{mol}{m^3}$ (separator). It is also can be referred to battery solid phase or electrolyte phase. We define two functions C_1 and C_3 which

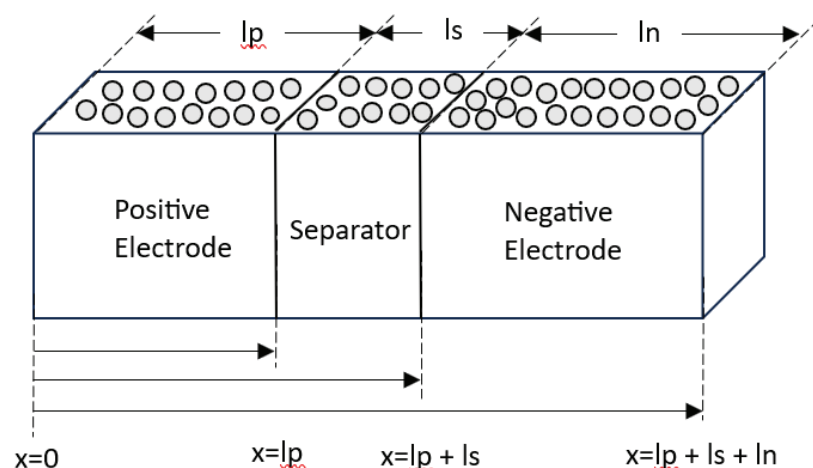


Figure 1: Lithium – ion battery cell structure (positive electrode, separator, and negative electrode)

are function of $C_{ave}^{cathode}$, C_{ave} , C_{ave}^{anode} variables and system overall parameters. I is defined as the applied battery cell current and it is related to the pore wall flux we get (J or j) [7]. The pore wall flux represents the local transfer rate of lithium ions between the solid active material and the liquid electrolyte. We define two functions $\xi_1 = \xi_1(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = C_1$ and $\xi_3 = \xi_3(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = C_3$.

$$\xi_1 = \xi_1(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = C_1 = \frac{3\varepsilon_p^4 C_{ave}^{cathode} (t - \tau_2) L_s + 6\varepsilon_s^4 L_p C_{ave} (t - \tau_1) - 2\varepsilon_s^4 L_p C_3}{3\varepsilon_p^4 L_s + 4\varepsilon_s^4 L_p} \quad (1)$$

$$\xi_1 = \xi_1(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = \frac{3\varepsilon_p^4 C_{ave}^{cathode} (t - \tau_2) L_s + 6\varepsilon_s^4 L_p C_{ave} (t - \tau_1) - 2\varepsilon_s^4 L_p \xi_3 (C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode})}{3\varepsilon_p^4 L_s + 4\varepsilon_s^4 L_p} \quad (2)$$

We define for simplicity some global system parameters: $\Xi_1 = 3\varepsilon_p^4 L_s$; $\Xi_2 = 6\varepsilon_s^4 L_p$; $\Xi_3 = -2\varepsilon_s^4 L_p$ and $\Xi_4 = 3\varepsilon_p^4 L_s + 4\varepsilon_s^4 L_p$. Then we get the reduced $\xi_1(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode})$ function:

$$\xi_1 = \xi_1(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = \frac{\Xi_1 C_{ave}^{cathode} (t - \tau_2) + \Xi_2 C_{ave} (t - \tau_1) + \Xi_3 \xi_3 (C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode})}{\Xi_4} \quad (3)$$

$$\xi_3 = \xi_3(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = C_3 = \frac{-2\varepsilon_s^4 L_n \varepsilon_p^4 L_s C_{ave}^{cathode} (t - \tau_2) + 4\varepsilon_s^8 L_n L_p C_{ave} (t - \tau_1) + 6\varepsilon_s^4 L_n L_s \varepsilon_p^4 C_{ave} (t - \tau_1) + 3\varepsilon_n^4 L_s^2 \varepsilon_p^4 C_{ave}^{anode} (t - \tau_3) + 4\varepsilon_s^4 L_s \varepsilon_s^4 L_p C_{ave}^{anode} (t - \tau_3)}{4\varepsilon_s^8 L_n L_p + 4\varepsilon_s^4 L_n L_s \varepsilon_p^4 + 3\varepsilon_n^4 L_s^2 \varepsilon_p^4 + 4\varepsilon_n^4 L_s \varepsilon_s^4 L_p} \quad (4)$$

$$\xi_3 = \xi_3(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = C_3 = \frac{-2\varepsilon_s^4 L_n \varepsilon_p^4 L_s C_{ave}^{cathode} (t - \tau_2) + [4\varepsilon_s^8 L_n L_p + 6\varepsilon_s^4 L_n L_s \varepsilon_p^4] C_{ave} (t - \tau_1) + [3\varepsilon_n^4 L_s^2 \varepsilon_p^4 + 4\varepsilon_s^4 L_s \varepsilon_s^4 L_p] C_{ave}^{anode} (t - \tau_3)}{4\varepsilon_s^8 L_n L_p + 4\varepsilon_s^4 L_n L_s \varepsilon_p^4 + 3\varepsilon_n^4 L_s^2 \varepsilon_p^4 + 4\varepsilon_n^4 L_s \varepsilon_s^4 L_p} \quad (5)$$

We define for simplicity some global system parameters: $\Xi_5 = -2\varepsilon_s^4 L_n \varepsilon_p^4 L_s$; $\Xi_6 = 4\varepsilon_s^8 L_n L_p + 6\varepsilon_s^4 L_n L_s \varepsilon_p^4$,

$$\Xi_7 = 3\varepsilon_n^4 L_s^2 \varepsilon_p^4 + 4\varepsilon_s^4 L_s \varepsilon_s^4 L_p; \Xi_8 = 4\varepsilon_s^8 L_n L_p + 4\varepsilon_s^4 L_n L_s \varepsilon_p^4 + 3\varepsilon_n^4 L_s^2 \varepsilon_p^4 + 4\varepsilon_n^4 L_s \varepsilon_s^4 L_p.$$

$$\xi_3 = \xi_3(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode}) = C_3 = \frac{\Xi_5 C_{ave}^{cathode} (t - \tau_2) + \Xi_6 C_{ave} (t - \tau_1) + \Xi_7 C_{ave}^{anode} (t - \tau_3)}{\Xi_8} \quad (6)$$

We can summery our system DDEs and additional related functions:

$$\frac{dC_{ave}}{dt} = \frac{-6\varepsilon_s^3 (2C_{ave} (t - \tau_1) - \xi_1 - \xi_3)}{L_s^2}; \frac{dC_{ave}^{cathode}}{dt} = \frac{3\varepsilon_p^3 (\xi_1 - C_{ave}^{cathode} (t - \tau_2))}{L_p^2} + J_p \text{ and}$$

$$\frac{dC_{ave}^{anode}}{dt} = \frac{3\varepsilon_n^3 (\xi_3 - C_{ave}^{anode} (t - \tau_3))(1 - L_p - L_s)}{L_n^3} + J_n; \xi_1 = \frac{\Xi_1 C_{ave}^{cathode} (t - \tau_2) + \Xi_2 C_{ave} (t - \tau_1) + \Xi_3 \xi_3}{\Xi_4}$$

$$\xi_3 = \frac{\Xi_5 C_{ave}^{cathode} (t - \tau_2) + \Xi_6 C_{ave} (t - \tau_1) + \Xi_7 C_{ave}^{anode} (t - \tau_3)}{\Xi_8}; \xi_3 = \frac{\Xi_5}{\Xi_8} C_{ave}^{cathode} (t - \tau_2) + \frac{\Xi_6}{\Xi_8} C_{ave} (t - \tau_1) + \frac{\Xi_7}{\Xi_8} C_{ave}^{anode} (t - \tau_3)$$

We define for simplicity additional global parameter: $\Upsilon_1 = \frac{\Xi_5}{\Xi_8}$; $\Upsilon_2 = \frac{\Xi_6}{\Xi_8}$; $\Upsilon_3 = \frac{\Xi_7}{\Xi_8}$ then

$$\xi_3 = \Upsilon_1 C_{ave}^{cathode} (t - \tau_2) + \Upsilon_2 C_{ave} (t - \tau_1) + \Upsilon_3 C_{ave}^{anode} (t - \tau_3) \quad (7)$$

$$\xi_1 = \frac{\Xi_1 C_{ave}^{cathode}(t-\tau_2) + \Xi_2 C_{ave}(t-\tau_1) + \Xi_3 \xi_3}{\Xi_4} = \frac{\Xi_1 C_{ave}^{cathode}(t-\tau_2) + \Xi_2 C_{ave}(t-\tau_1) + \Xi_3 \xi_3}{\Xi_4} \quad (8)$$

$$\xi_1 = \frac{[\Xi_1 + \frac{\Xi_3 \Xi_5}{\Xi_8}] C_{ave}^{cathode}(t-\tau_2) + [\Xi_2 + \frac{\Xi_3 \Xi_6}{\Xi_8}] C_{ave}(t-\tau_1) + \frac{\Xi_3 \Xi_7}{\Xi_8} C_{ave}^{anode}(t-\tau_3)}{\Xi_4} \quad (9)$$

$$\xi_1 = \frac{1}{\Xi_4} [\Xi_1 + \frac{\Xi_3 \Xi_5}{\Xi_8}] C_{ave}^{cathode}(t-\tau_2) + \frac{1}{\Xi_4} [\Xi_2 + \frac{\Xi_3 \Xi_6}{\Xi_8}] C_{ave}(t-\tau_1) + \frac{\Xi_3 \Xi_7}{\Xi_8 \Xi_4} C_{ave}^{anode}(t-\tau_3) \quad (10)$$

We define for simplicity additional global parameter: $\Upsilon_4 = \frac{1}{\Xi_4} [\Xi_1 + \frac{\Xi_3 \Xi_5}{\Xi_8}]$; $\Upsilon_5 = \frac{1}{\Xi_4} [\Xi_2 + \frac{\Xi_3 \Xi_6}{\Xi_8}]$ and

$$\Upsilon_6 = \frac{\Xi_3 \Xi_7}{\Xi_8 \Xi_4}, \text{ then } \xi_1 = \Upsilon_4 C_{ave}^{cathode}(t-\tau_2) + \Upsilon_5 C_{ave}(t-\tau_1) + \Upsilon_6 C_{ave}^{anode}(t-\tau_3).$$

We can summary our $\xi_1(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode})$ and $\xi_3(C_{ave}^{cathode}, C_{ave}, C_{ave}^{anode})$ equations:

$$\xi_3 = \Upsilon_1 C_{ave}^{cathode}(t-\tau_2) + \Upsilon_2 C_{ave}(t-\tau_1) + \Upsilon_3 C_{ave}^{anode}(t-\tau_3) \quad (11)$$

$$\xi_1 = \Upsilon_4 C_{ave}^{cathode}(t-\tau_2) + \Upsilon_5 C_{ave}(t-\tau_1) + \Upsilon_6 C_{ave}^{anode}(t-\tau_3) \quad (12)$$

At fixed points (system equilibrium points) $\frac{dC_{ave}}{dt} = 0$; $\frac{dC_{ave}^{cathode}}{dt} = 0$; $\frac{dC_{ave}^{anode}}{dt} = 0$, $C_{ave}(t-\tau_1) \approx C_{ave}(t)$

$\forall t \gg \tau_1$; $\tau_1 \in \mathbb{R}$, $C_{ave}^{cathode}(t-\tau_2) \approx C_{ave}^{cathode}(t) \forall t \gg \tau_2$; $\tau_2 \in \mathbb{R}$, and $C_{ave}^{anode}(t-\tau_3) \approx C_{ave}^{anode}(t)$

$\forall t \gg \tau_3$; $\tau_3 \in \mathbb{R}$; $\tau_1, \tau_2, \tau_3 > 0$. At fixed points $\xi_3^* = \Upsilon_1 [C_{ave}^{cathode}]^* + \Upsilon_2 [C_{ave}]^* + \Upsilon_3 [C_{ave}^{anode}]^*$ and

$$\xi_1^* = \Upsilon_4 [C_{ave}^{cathode}]^* + \Upsilon_5 [C_{ave}]^* + \Upsilon_6 [C_{ave}^{anode}]^*.$$

$$\frac{dC_{ave}}{dt} = 0 \Rightarrow \frac{-6\epsilon_s^3 (2[C_{ave}]^* - \xi_1^* - \xi_3^*)}{L_s^2} = 0; \frac{dC_{ave}^{cathode}}{dt} = 0 \Rightarrow \frac{3\epsilon_p^3 (\xi_1^* - [C_{ave}^{cathode}]^*)}{L_p^2} + J_p = 0 \quad (13)$$

$$\frac{dC_{ave}^{anode}}{dt} = 0 \Rightarrow \frac{3\epsilon_n^3 (\xi_3^* - [C_{ave}^{anode}]^*) (1 - L_p - L_s)}{L_n^3} + J_n = 0 \quad (14)$$

First fixed point equation (equilibrium points):

$$\frac{-6\epsilon_s^3 (2[C_{ave}]^* - \xi_1^* - \xi_3^*)}{L_s^2} = 0 \Rightarrow \frac{-\{ \Upsilon_1 [C_{ave}^{cathode}]^* + \Upsilon_2 [C_{ave}]^* + \Upsilon_3 [C_{ave}^{anode}]^* \}}{L_s^2} = 0 \quad (15)$$

$$\frac{-6\epsilon_s^3 (2[C_{ave}]^* - \xi_1^* - \xi_3^*)}{L_s^2} = 0 \Rightarrow \frac{-(\Upsilon_6 + \Upsilon_3) [C_{ave}^{anode}]^*}{L_s^2} = 0 \quad (16)$$

$$L_s^2 \neq 0; -6\epsilon_s^3 \neq 0; (2 - \Upsilon_5 - \Upsilon_2) [C_{ave}]^* - (\Upsilon_4 + \Upsilon_1) [C_{ave}^{cathode}]^* - (\Upsilon_6 + \Upsilon_3) [C_{ave}^{anode}]^* = 0 \quad (17)$$

Second fixed point equation (equilibrium points):

$$\frac{3\varepsilon_p^3(\xi_1^* - [C_{ave}^{cathode}]^*)}{L_p^2} + J_p = 0 \Rightarrow \frac{3\varepsilon_p^3(\{\Upsilon_4[C_{ave}^{cathode}]^* + \Upsilon_5[C_{ave}]^* + \Upsilon_6[C_{ave}^{anode}]^*\} - [C_{ave}^{cathode}]^*)}{L_p^2} + J_p = 0 \quad (18)$$

$$\frac{3\varepsilon_p^3(\xi_1^* - [C_{ave}^{cathode}]^*)}{L_p^2} + J_p = 0 \Rightarrow \frac{3\varepsilon_p^3\{(\Upsilon_4 - 1)[C_{ave}^{cathode}]^* + \Upsilon_5[C_{ave}]^* + \Upsilon_6[C_{ave}^{anode}]^*\}}{L_p^2} + J_p = 0 \quad (19)$$

$$(\Upsilon_4 - 1)[C_{ave}^{cathode}]^* + \Upsilon_5[C_{ave}]^* + \Upsilon_6[C_{ave}^{anode}]^* + \frac{J_p L_p^2}{3\varepsilon_p^3} = 0 \quad (20)$$

Third fixed point equation (equilibrium points):

$$\frac{3\varepsilon_n^3(\xi_3^* - [C_{ave}^{anode}]^*)(1 - L_p - L_s)}{L_n^3} + J_n = 0 \Rightarrow \frac{3\varepsilon_n^3(\{\Upsilon_1[C_{ave}^{cathode}]^* + \Upsilon_2[C_{ave}]^* + \Upsilon_3[C_{ave}^{anode}]^*\} - [C_{ave}^{anode}]^*)(1 - L_p - L_s)}{L_n^3} + J_n = 0 \quad (21)$$

$$\frac{3\varepsilon_n^3(\xi_3^* - [C_{ave}^{anode}]^*)(1 - L_p - L_s)}{L_n^3} + J_n = 0 \Rightarrow \frac{3\varepsilon_n^3\{(\Upsilon_1 - 1)[C_{ave}^{cathode}]^* + \Upsilon_2[C_{ave}]^* + \Upsilon_3[C_{ave}^{anode}]^*\}(1 - L_p - L_s)}{L_n^3} + J_n = 0 \quad (22)$$

$$\Upsilon_1[C_{ave}^{cathode}]^* + \Upsilon_2[C_{ave}]^* + (\Upsilon_3 - 1)[C_{ave}^{anode}]^* + \frac{J_n L_n^3}{3\varepsilon_n^3(1 - L_p - L_s)} = 0 \quad (23)$$

We can summery our system three fixed points equations:

$$(2 - \Upsilon_5 - \Upsilon_2)[C_{ave}]^* - (\Upsilon_4 + \Upsilon_1)[C_{ave}^{cathode}]^* - (\Upsilon_6 + \Upsilon_3)[C_{ave}^{anode}]^* = 0 \quad (24)$$

$$(\Upsilon_4 - 1)[C_{ave}^{cathode}]^* + \Upsilon_5[C_{ave}]^* + \Upsilon_6[C_{ave}^{anode}]^* + \frac{J_p L_p^2}{3\varepsilon_p^3} = 0 \quad (25)$$

$$\Upsilon_1[C_{ave}^{cathode}]^* + \Upsilon_2[C_{ave}]^* + (\Upsilon_3 - 1)[C_{ave}^{anode}]^* + \frac{J_n L_n^3}{3\varepsilon_n^3(1 - L_p - L_s)} = 0 \quad (26)$$

We get system fixed points: $E^*([C_{ave}]^*, [C_{ave}^{cathode}]^*, [C_{ave}^{anode}]^*)$.

The Lit – ion battery delay differential equations (DDEs) model can be compared to the traditional reduced order Lit – ion battery model. The Lit – ion reduced – order model (ROM) simplifies the model's partial differential equations (PDEs) in terms of solid diffusion and electrolyte migration. It is done into lower – dimensional approximations approach. It is applicable to single particle model (SPM) and SPM with electrode. We can implement it also to Galerkin projection methods. The Galerkin projection method is implemented as a mathematical technique in our case and use to convert continuous differential equations into a smaller and define – dimensional system equations. It is done by projecting the residual orthogonally elements onto a subspace of test function. Compare to Lit – ion reduced order model (ROM) the delay differential equations (DDEs) Lit – ion battery model uses time – delayed states such as diffusion lags, currents, or past voltages to account for thermal incubation, transport time or delayed internal feedback loops.

Lithium-ion batteries solid-phase diffusion Approximations, transient response, electrolyte concentrations characteristic equation $\tau_1=\tau$, $\tau_2=\tau_3=0$

We get the expressions for $\xi_1 = \xi_1(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0)$ and $\xi_3 = \xi_3(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0)$, then

$$\xi_3(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = Y_1 C_{ave}^{cathode}(t) + Y_2 C_{ave}(t - \tau) + Y_3 C_{ave}^{anode}(t) \quad (27)$$

$$\xi_1(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = Y_4 C_{ave}^{cathode}(t) + Y_5 C_{ave}(t - \tau) + Y_6 C_{ave}^{anode}(t) \quad (28)$$

The standard local stability analysis about any one of the equilibrium points of Lithium-ion batteries solid-phase electrolyte concentrations system consists in adding to coordinate $[C_{ave}, C_{ave}^{cathode}, C_{ave}^{anode}]$ arbitrarily small increments of exponential form $[c_{ave}, c_{ave}^{cathode}, c_{ave}^{anode}]e^{\lambda t}$ and retaining the first order term in $c_{ave}, c_{ave}^{cathode}, c_{ave}^{anode}$. The system of three homogeneous equations leads to a polynomial characteristic equation in the eigenvalue λ [8][9]. The Lithium-ion batteries solid-phase electrolyte concentrations system fixed values with arbitrarily small increments of exponential form $[c_{ave}, c_{ave}^{cathode}, c_{ave}^{anode}]e^{\lambda t}$ are $k = 1$ (first fixed point), $k = 2$ (second fixed point), $k = 3$ (third fixed point), ...

$$C_{ave}(t) = [C_{ave}]^* + c_{ave}e^{\lambda t}; C_{ave}^{cathode}(t) = [C_{ave}^{cathode}]^* + c_{ave}^{cathode}e^{\lambda t}; C_{ave}^{anode}(t) = [C_{ave}^{anode}]^* + c_{ave}^{anode}e^{\lambda t} \quad (29)$$

$$C_{ave}(t - \tau) = [C_{ave}]^* + c_{ave}e^{\lambda(t-\tau)}, C_{ave}^{cathode}(t - \tau) = [C_{ave}^{cathode}]^* + c_{ave}^{cathode}e^{\lambda(t-\tau)}, C_{ave}^{anode}(t - \tau) = [C_{ave}^{anode}]^* + c_{ave}^{anode}e^{\lambda(t-\tau)} \quad (30)$$

$$\xi_3(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = Y_1([C_{ave}^{cathode}]^* + c_{ave}^{cathode}e^{\lambda t}) + Y_2([C_{ave}]^* + c_{ave}e^{\lambda(t-\tau)}) + Y_3([C_{ave}^{anode}]^* + c_{ave}^{anode}e^{\lambda t}) \quad (31)$$

$$\xi_3(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = Y_1[C_{ave}^{cathode}]^* + Y_2[C_{ave}]^* + Y_3[C_{ave}^{anode}]^* + Y_1c_{ave}^{cathode}e^{\lambda t} + Y_2c_{ave}e^{\lambda(t-\tau)} + Y_3c_{ave}^{anode}e^{\lambda t} \quad (32)$$

We define ξ_3^* fixed expression as $\xi_3^* = Y_1[C_{ave}^{cathode}]^* + Y_2[C_{ave}]^* + Y_3[C_{ave}^{anode}]^*$ then $\xi_3(t, \tau_1 = \tau, \tau_2 = \tau_3 = 0)$ is $\xi_3(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = \xi_3^* + Y_1c_{ave}^{cathode}e^{\lambda t} + Y_2c_{ave}e^{\lambda(t-\tau)} + Y_3c_{ave}^{anode}e^{\lambda t}$.

$$\xi_1(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = Y_4([C_{ave}^{cathode}]^* + c_{ave}^{cathode}e^{\lambda t}) + Y_5([C_{ave}]^* + c_{ave}e^{\lambda(t-\tau)}) + Y_6([C_{ave}^{anode}]^* + c_{ave}^{anode}e^{\lambda t}) \quad (33)$$

$$\xi_1(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = Y_4[C_{ave}^{cathode}]^* + Y_5[C_{ave}]^* + Y_6[C_{ave}^{anode}]^* + Y_4c_{ave}^{cathode}e^{\lambda t} + Y_5c_{ave}e^{\lambda(t-\tau)} + Y_6c_{ave}^{anode}e^{\lambda t} \quad (34)$$

We define ξ_1^* fixed expression as $\xi_1^* = Y_4[C_{ave}^{cathode}]^* + Y_5[C_{ave}]^* + Y_6[C_{ave}^{anode}]^*$ then $\xi_1(t, \tau_1 = \tau, \tau_2 = \tau_3 = 0)$

is $\xi_1(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = \xi_1^* + Y_4c_{ave}^{cathode}e^{\lambda t} + Y_5c_{ave}e^{\lambda(t-\tau)} + Y_6c_{ave}^{anode}e^{\lambda t}$.

We choose the above expressions for our $C_{ave}^{cathode}(t), C_{ave}(t), C_{ave}^{anode}(t)$ as small displacement $c_{ave}, c_{ave}^{cathode}, c_{ave}^{anode}$ from the system fixed points at time $t = 0$ ($C_{ave} = C_{ave}(t), C_{ave}^{cathode} = C_{ave}^{cathode}(t), C_{ave}^{anode} = C_{ave}^{anode}(t)$).

$$\frac{dC_{ave}(t)}{dt} = c_{ave}\lambda e^{\lambda t}; \frac{dC_{ave}^{cathode}(t)}{dt} = c_{ave}^{cathode}\lambda e^{\lambda t}; \frac{dC_{ave}^{anode}(t)}{dt} = c_{ave}^{anode}\lambda e^{\lambda t} \quad (35)$$

$$\frac{dC_{ave}(t - \tau)}{dt} = c_{ave}\lambda e^{\lambda(t-\tau)}, \frac{dC_{ave}^{cathode}(t - \tau)}{dt} = c_{ave}^{cathode}\lambda e^{\lambda(t-\tau)}, \frac{dC_{ave}^{anode}(t - \tau)}{dt} = c_{ave}^{anode}\lambda e^{\lambda(t-\tau)} \quad (36)$$

Under our specific delay parameters value ($\tau_1 = \tau, \tau_2 = 0, \tau_3 = 0$), we get the following system delay differential equations (DDEs):

$$\frac{dC_{ave}}{dt} = \frac{-6\varepsilon_s^3(2C_{ave}(t-\tau) - \xi_1 - \xi_3)}{L_s^2}; \frac{dC_{ave}^{cathode}}{dt} = \frac{3\varepsilon_p^3(\xi_1 - C_{ave}^{cathode}(t))}{L_p^2} + J_p \text{ and}$$

$$\frac{dC_{ave}^{anode}}{dt} = \frac{3\varepsilon_n^3(\xi_3 - C_{ave}^{anode}(t))(1 - L_p - L_s)}{L_n^3} + J_n. \text{ The above } \xi_1, \xi_3 \text{ expressions are related to our specific case (}$$

$$\tau_1 = \tau, \tau_2 = 0, \tau_3 = 0), \xi_1(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) \text{ and } \xi_3(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0).$$

$$\begin{aligned} & -6\varepsilon_s^3\{2([C_{ave}]^* + c_{ave}e^{\lambda(t-\tau)}) \\ & -(\xi_1^* + \Upsilon_4c_{ave}^{cathode}e^{\lambda t} + \Upsilon_5c_{ave}e^{\lambda(t-\tau)} + \Upsilon_6c_{ave}^{anode}e^{\lambda t})\} \\ \frac{dC_{ave}}{dt} = \frac{-6\varepsilon_s^3(2C_{ave}(t-\tau) - \xi_1 - \xi_3)}{L_s^2} \Rightarrow c_{ave}\lambda e^{\lambda t} = \frac{-(\xi_3^* + \Upsilon_1c_{ave}^{cathode}e^{\lambda t} + \Upsilon_2c_{ave}e^{\lambda(t-\tau)} + \Upsilon_3c_{ave}^{anode}e^{\lambda t})}{L_s^2} \end{aligned} \quad (37)$$

$$\begin{aligned} & -6\varepsilon_s^3\{2([C_{ave}]^* + c_{ave}e^{\lambda(t-\tau)}) - (\xi_1^* + \Upsilon_4c_{ave}^{cathode}e^{\lambda t} + \Upsilon_5c_{ave}e^{\lambda(t-\tau)} + \Upsilon_6c_{ave}^{anode}e^{\lambda t}) \\ & -(\xi_3^* + \Upsilon_1c_{ave}^{cathode}e^{\lambda t} + \Upsilon_2c_{ave}e^{\lambda(t-\tau)} + \Upsilon_3c_{ave}^{anode}e^{\lambda t})\} = -6\varepsilon_s^32[C_{ave}]^* - 6\varepsilon_s^32c_{ave}e^{\lambda(t-\tau)} \\ & + 6\varepsilon_s^3\xi_1^* + 6\varepsilon_s^3\Upsilon_4c_{ave}^{cathode}e^{\lambda t} + 6\varepsilon_s^3\Upsilon_5c_{ave}e^{\lambda(t-\tau)} + 6\varepsilon_s^3\Upsilon_6c_{ave}^{anode}e^{\lambda t} \\ & + 6\varepsilon_s^3\xi_3^* + 6\varepsilon_s^3\Upsilon_1c_{ave}^{cathode}e^{\lambda t} + 6\varepsilon_s^3\Upsilon_2c_{ave}e^{\lambda(t-\tau)} + 6\varepsilon_s^3\Upsilon_3c_{ave}^{anode}e^{\lambda t} \end{aligned} \quad (38)$$

$$\begin{aligned} & -6\varepsilon_s^32[C_{ave}]^* - 6\varepsilon_s^32c_{ave}e^{\lambda(t-\tau)} + 6\varepsilon_s^3\xi_1^* + 6\varepsilon_s^3\Upsilon_4c_{ave}^{cathode}e^{\lambda t} + 6\varepsilon_s^3\Upsilon_5c_{ave}e^{\lambda(t-\tau)} \\ & + 6\varepsilon_s^3\Upsilon_6c_{ave}^{anode}e^{\lambda t} + 6\varepsilon_s^3\xi_3^* + 6\varepsilon_s^3\Upsilon_1c_{ave}^{cathode}e^{\lambda t} + 6\varepsilon_s^3\Upsilon_2c_{ave}e^{\lambda(t-\tau)} + 6\varepsilon_s^3\Upsilon_3c_{ave}^{anode}e^{\lambda t} \\ & = -6(\varepsilon_s^32[C_{ave}]^* - \varepsilon_s^3\xi_1^* - \varepsilon_s^3\xi_3^*) + 6\{\varepsilon_s^3[-2 + \Upsilon_5 + \Upsilon_2]c_{ave}e^{-\lambda\tau} \\ & + \varepsilon_s^3[\Upsilon_4 + \Upsilon_1]c_{ave}^{cathode} + \varepsilon_s^3[\Upsilon_6 + \Upsilon_3]c_{ave}^{anode}\}e^{\lambda t} \end{aligned} \quad (39)$$

We get the following first eigenvalue function:

$$\begin{aligned} & -6(\varepsilon_s^32[C_{ave}]^* - \varepsilon_s^3\xi_1^* - \varepsilon_s^3\xi_3^*) + 6\{\varepsilon_s^3[-2 + \Upsilon_5 + \Upsilon_2]c_{ave}e^{-\lambda\tau} \\ & + \varepsilon_s^3[\Upsilon_4 + \Upsilon_1]c_{ave}^{cathode} + \varepsilon_s^3[\Upsilon_6 + \Upsilon_3]c_{ave}^{anode}\}e^{\lambda t} \\ c_{ave}\lambda e^{\lambda t} = \frac{L_s^2}{} \end{aligned} \quad (40)$$

$$\begin{aligned} & 6\{\varepsilon_s^3[-2 + \Upsilon_5 + \Upsilon_2]c_{ave}e^{-\lambda\tau} \\ c_{ave}\lambda e^{\lambda t} = \frac{-6\varepsilon_s^3(2[C_{ave}]^* - \xi_1^* - \xi_3^*)}{L_s^2} + \frac{+\varepsilon_s^3[\Upsilon_4 + \Upsilon_1]c_{ave}^{cathode} + \varepsilon_s^3[\Upsilon_6 + \Upsilon_3]c_{ave}^{anode}}{L_s^2}\}e^{\lambda t} \end{aligned} \quad (41)$$

Since $\frac{dC_{ave}}{dt} = \frac{-6\varepsilon_s^3(2C_{ave}(t-\tau) - \xi_1 - \xi_3)}{L_s^2}$, at fixed point $(\frac{dC_{ave}}{dt} = 0; \lim_{t \rightarrow \infty, t \gg \tau} C_{ave}(t-\tau) = C_{ave}(t))$ exists $\frac{-6\varepsilon_s^3(2[C_{ave}]^* - \xi_1^* - \xi_3^*)}{L_s^2} = 0$ and the first eigenvalue equation is

$$c_{ave}\lambda = \frac{6\{\varepsilon_s^3[-2 + \Upsilon_5 + \Upsilon_2]c_{ave}e^{-\lambda\tau} + \varepsilon_s^3[\Upsilon_4 + \Upsilon_1]c_{ave}^{cathode} + \varepsilon_s^3[\Upsilon_6 + \Upsilon_3]c_{ave}^{anode}\}}{L_s^2} \quad (42)$$

$$(\frac{6\varepsilon_s^3[-2 + \Upsilon_5 + \Upsilon_2]e^{-\lambda\tau}}{L_s^2} - \lambda)c_{ave} + \frac{6\varepsilon_s^3[\Upsilon_4 + \Upsilon_1]}{L_s^2}c_{ave}^{cathode} + \frac{6\varepsilon_s^3[\Upsilon_6 + \Upsilon_3]}{L_s^2}c_{ave}^{anode} = 0 \quad (43)$$

We define for simplicity system global parameters: $\Xi_1 = \frac{6\varepsilon_s^3[-2 + \Upsilon_5 + \Upsilon_2]}{L_s^2}$; $\Xi_2 = \frac{6\varepsilon_s^3[\Upsilon_4 + \Upsilon_1]}{L_s^2}$
 $\Xi_3 = \frac{6\varepsilon_s^3[\Upsilon_6 + \Upsilon_3]}{L_s^2}$ then the first eigenvalue equation is $(\Xi_1 e^{-\lambda\tau} - \lambda)c_{ave} + \Xi_2 c_{ave}^{cathode} + \Xi_3 c_{ave}^{anode} = 0$.

$$\frac{dC_{ave}^{cathode}}{dt} = \frac{3\varepsilon_p^3(\xi_1 - C_{ave}^{cathode}(t))}{L_p^2} + J_p \Rightarrow c_{ave}^{cathode} \lambda e^{\lambda t} = \frac{3\varepsilon_p^3[(\xi_1^* + \Upsilon_4 c_{ave}^{cathode} e^{\lambda t} + \Upsilon_5 c_{ave} e^{\lambda(t-\tau)} + \Upsilon_6 c_{ave}^{anode} e^{\lambda t}) - ([C_{ave}^{cathode}]^* + c_{ave}^{cathode} e^{\lambda t})]}{L_p^2} + J_p \quad (44)$$

$$3\varepsilon_p^3[(\xi_1^* + \Upsilon_4 c_{ave}^{cathode} e^{\lambda t} + \Upsilon_5 c_{ave} e^{\lambda(t-\tau)} + \Upsilon_6 c_{ave}^{anode} e^{\lambda t}) - ([C_{ave}^{cathode}]^* + c_{ave}^{cathode} e^{\lambda t})] = 3\varepsilon_p^3 \xi_1^* + 3\varepsilon_p^3 \Upsilon_4 c_{ave}^{cathode} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_5 c_{ave} e^{\lambda(t-\tau)} + 3\varepsilon_p^3 \Upsilon_6 c_{ave}^{anode} e^{\lambda t} - 3\varepsilon_p^3 [C_{ave}^{cathode}]^* - 3\varepsilon_p^3 c_{ave}^{cathode} e^{\lambda t} \quad (45)$$

$$3\varepsilon_p^3 \xi_1^* + 3\varepsilon_p^3 \Upsilon_4 c_{ave}^{cathode} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_5 c_{ave} e^{\lambda(t-\tau)} + 3\varepsilon_p^3 \Upsilon_6 c_{ave}^{anode} e^{\lambda t} - 3\varepsilon_p^3 [C_{ave}^{cathode}]^* - 3\varepsilon_p^3 c_{ave}^{cathode} e^{\lambda t} = 3\varepsilon_p^3 (\xi_1^* - [C_{ave}^{cathode}]^*) + 3\varepsilon_p^3 \Upsilon_5 c_{ave} e^{\lambda(t-\tau)} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{ave}^{cathode} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{ave}^{anode} e^{\lambda t} \quad (46)$$

We get the following second eigenvalue function:

$$c_{ave}^{cathode} \lambda e^{\lambda t} = \frac{3\varepsilon_p^3(\xi_1^* - [C_{ave}^{cathode}]^*) + 3\varepsilon_p^3 \Upsilon_5 c_{ave} e^{\lambda(t-\tau)} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{ave}^{cathode} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{ave}^{anode} e^{\lambda t}}{L_p^2} + J_p \quad (47)$$

$$c_{ave}^{cathode} \lambda e^{\lambda t} = \frac{3\varepsilon_p^3(\xi_1^* - [C_{ave}^{cathode}]^*)}{L_p^2} + J_p + \frac{3\varepsilon_p^3 \Upsilon_5 c_{ave} e^{\lambda(t-\tau)} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{ave}^{cathode} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{ave}^{anode} e^{\lambda t}}{L_p^2} \quad (48)$$

Since $\frac{dC_{ave}^{cathode}}{dt} = \frac{3\varepsilon_p^3(\xi_1 - C_{ave}^{cathode}(t))}{L_p^2} + J_p$, at fixed point ($\frac{dC_{ave}^{cathode}}{dt} = 0$) exists

$$\frac{3\varepsilon_p^3(\xi_1^* - [C_{ave}^{cathode}]^*)}{L_p^2} + J_p = 0 \text{ and the second eigenvalue equation is}$$

$$c_{ave}^{cathode} \lambda = \frac{3\varepsilon_p^3 \Upsilon_5 c_{ave} e^{-\lambda\tau} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{ave}^{cathode} + 3\varepsilon_p^3 \Upsilon_6 c_{ave}^{anode}}{L_p^2} \quad (49)$$

$$\frac{3\varepsilon_p^3 \Upsilon_5 e^{-\lambda\tau}}{L_p^2} c_{ave} + \left(\frac{3\varepsilon_p^3 (\Upsilon_4 - 1)}{L_p^2} - \lambda \right) c_{ave}^{cathode} + \frac{3\varepsilon_p^3 \Upsilon_6}{L_p^2} c_{ave}^{anode} = 0 \quad (50)$$

We define for simplicity system global parameters: $\Xi_4 = \frac{3\varepsilon_p^3 \Upsilon_5}{L_p^2}$; $\Xi_5 = \frac{3\varepsilon_p^3 (\Upsilon_4 - 1)}{L_p^2}$; $\Xi_6 = \frac{3\varepsilon_p^3 \Upsilon_6}{L_p^2}$ then the second

eigenvalue equation is $\Xi_4 e^{-\lambda\tau} c_{ave} + (\Xi_5 - \lambda) c_{ave}^{cathode} + \Xi_6 c_{ave}^{anode} = 0$.

$$\frac{dC_{ave}^{anode}}{dt} = \frac{3\varepsilon_n^3(\xi_3 - C_{ave}^{anode}(t))(1 - L_p - L_s)}{L_n^3} + J_n \Rightarrow c_{ave}^{anode} \lambda e^{\lambda t} = \frac{3\varepsilon_n^3[(\xi_3^* + \Upsilon_1 c_{ave}^{cathode} e^{\lambda t} + \Upsilon_2 c_{ave} e^{\lambda(t-\tau)} + \Upsilon_3 c_{ave}^{anode} e^{\lambda t}) - ([C_{ave}^{anode}]^* + c_{ave}^{anode} e^{\lambda t})](1 - L_p - L_s)}{L_n^3} + J_n \quad (51)$$

$$\begin{aligned}
 & 3\varepsilon_n^3([\xi_3^* + \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_2 c_{\text{ave}} e^{\lambda(t-\tau)} + \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda t}] - [C_{\text{ave}}^{\text{anode}}]^* + c_{\text{ave}}^{\text{anode}} e^{\lambda t}](1-L_p-L_s) \\
 & = 3\varepsilon_n^3 \xi_3^*(1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda(t-\tau)} (1-L_p-L_s) \\
 & + 3\varepsilon_n^3 \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda t} (1-L_p-L_s) - 3\varepsilon_n^3 [C_{\text{ave}}^{\text{anode}}]^* (1-L_p-L_s) - 3\varepsilon_n^3 c_{\text{ave}}^{\text{anode}} e^{\lambda t} (1-L_p-L_s)
 \end{aligned} \tag{52}$$

$$\begin{aligned}
 & 3\varepsilon_n^3 \xi_3^*(1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda(t-\tau)} (1-L_p-L_s) \\
 & + 3\varepsilon_n^3 \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda t} (1-L_p-L_s) - 3\varepsilon_n^3 [C_{\text{ave}}^{\text{anode}}]^* (1-L_p-L_s) - 3\varepsilon_n^3 c_{\text{ave}}^{\text{anode}} e^{\lambda t} (1-L_p-L_s) \\
 & = 3\varepsilon_n^3 (\xi_3^* - [C_{\text{ave}}^{\text{anode}}]^*) (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda(t-\tau)} (1-L_p-L_s) \\
 & + 3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 [\Upsilon_3 - 1] c_{\text{ave}}^{\text{anode}} e^{\lambda t} (1-L_p-L_s)
 \end{aligned} \tag{53}$$

We get the following third eigenvalue function:

$$\begin{aligned}
 & 3\varepsilon_n^3 (\xi_3^* - [C_{\text{ave}}^{\text{anode}}]^*) (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda(t-\tau)} (1-L_p-L_s) \\
 c_{\text{ave}}^{\text{anode}} \lambda e^{\lambda t} = & \frac{+3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 [\Upsilon_3 - 1] c_{\text{ave}}^{\text{anode}} e^{\lambda t} (1-L_p-L_s)}{L_n^3} + J_n
 \end{aligned} \tag{54}$$

$$\begin{aligned}
 c_{\text{ave}}^{\text{anode}} \lambda e^{\lambda t} = & \frac{3\varepsilon_n^3 (\xi_3^* - [C_{\text{ave}}^{\text{anode}}]^*) (1-L_p-L_s)}{L_n^3} + J_n + \frac{3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda(t-\tau)} (1-L_p-L_s)}{L_n^3} \\
 & + \frac{3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s)}{L_n^3} + \frac{3\varepsilon_n^3 [\Upsilon_3 - 1] c_{\text{ave}}^{\text{anode}} e^{\lambda t} (1-L_p-L_s)}{L_n^3}
 \end{aligned} \tag{55}$$

Since $\frac{dC_{\text{ave}}^{\text{anode}}}{dt} = \frac{3\varepsilon_n^3 (\xi_3 - C_{\text{ave}}^{\text{anode}}(t)) (1-L_p-L_s)}{L_n^3} + J_n$, at fixed point ($\frac{dC_{\text{ave}}^{\text{anode}}}{dt} = 0$) exists

$$\begin{aligned}
 & \frac{3\varepsilon_n^3 (\xi_3^* - [C_{\text{ave}}^{\text{anode}}]^*) (1-L_p-L_s)}{L_n^3} + J_n = 0 \text{ and the third eigenvalue equation [10][11] is} \\
 c_{\text{ave}}^{\text{anode}} \lambda = & \frac{3\varepsilon_n^3 \Upsilon_2 (1-L_p-L_s)}{L_n^3} e^{-\lambda \tau} c_{\text{ave}} + \frac{3\varepsilon_n^3 \Upsilon_1 (1-L_p-L_s)}{L_n^3} c_{\text{ave}}^{\text{cathode}} + \frac{3\varepsilon_n^3 [\Upsilon_3 - 1] (1-L_p-L_s)}{L_n^3} c_{\text{ave}}^{\text{anode}}
 \end{aligned} \tag{56}$$

We define for simplicity system global parameters: $\Xi_7 = \frac{3\varepsilon_n^3 \Upsilon_2 (1-L_p-L_s)}{L_n^3}$; $\Xi_8 = \frac{3\varepsilon_n^3 \Upsilon_1 (1-L_p-L_s)}{L_n^3}$

$$\Xi_9 = \frac{3\varepsilon_n^3 [\Upsilon_3 - 1] (1-L_p-L_s)}{L_n^3} \text{ then the third eigenvalue equation is } \Xi_7 e^{-\lambda \tau} c_{\text{ave}} + \Xi_8 c_{\text{ave}}^{\text{cathode}} + (\Xi_9 - \lambda) c_{\text{ave}}^{\text{anode}} = 0.$$

We can summary our system three eigenvalues equation with delay parameter (τ):

$$\text{First eigenvalue equation: } (\Xi_1 e^{-\lambda \tau} - \lambda) c_{\text{ave}} + \Xi_2 c_{\text{ave}}^{\text{cathode}} + \Xi_3 c_{\text{ave}}^{\text{anode}} = 0$$

$$\text{Second eigenvalue equation: } \Xi_4 e^{-\lambda \tau} c_{\text{ave}} + (\Xi_5 - \lambda) c_{\text{ave}}^{\text{cathode}} + \Xi_6 c_{\text{ave}}^{\text{anode}} = 0$$

$$\text{Third eigenvalue equation: } \Xi_7 e^{-\lambda \tau} c_{\text{ave}} + \Xi_8 c_{\text{ave}}^{\text{cathode}} + (\Xi_9 - \lambda) c_{\text{ave}}^{\text{anode}} = 0$$

$$\text{The eigenvalue matrix we get is } \begin{pmatrix} (\Xi_1 e^{-\lambda \tau} - \lambda) & \Xi_2 & \Xi_3 \\ \Xi_4 e^{-\lambda \tau} & (\Xi_5 - \lambda) & \Xi_6 \\ \Xi_7 e^{-\lambda \tau} & \Xi_8 & (\Xi_9 - \lambda) \end{pmatrix} \begin{pmatrix} c_{\text{ave}} \\ c_{\text{ave}}^{\text{cathode}} \\ c_{\text{ave}}^{\text{anode}} \end{pmatrix} = 0$$

$$A - \lambda I = \begin{pmatrix} (\Xi_1 e^{-\lambda \tau} - \lambda) & \Xi_2 & \Xi_3 \\ \Xi_4 e^{-\lambda \tau} & (\Xi_5 - \lambda) & \Xi_6 \\ \Xi_7 e^{-\lambda \tau} & \Xi_8 & (\Xi_9 - \lambda) \end{pmatrix}; \det(A - \lambda I) = 0 \quad (57)$$

$$\det(A - \lambda I) = (\Xi_1 e^{-\lambda \tau} - \lambda) \det \begin{pmatrix} (\Xi_5 - \lambda) & \Xi_6 \\ \Xi_8 & (\Xi_9 - \lambda) \end{pmatrix} - \Xi_2 \det \begin{pmatrix} \Xi_4 e^{-\lambda \tau} & \Xi_6 \\ \Xi_7 e^{-\lambda \tau} & (\Xi_9 - \lambda) \end{pmatrix} \\ + \Xi_3 \det \begin{pmatrix} \Xi_4 e^{-\lambda \tau} & (\Xi_5 - \lambda) \\ \Xi_7 e^{-\lambda \tau} & \Xi_8 \end{pmatrix} \quad (58)$$

$$\det(A - \lambda I) = (\Xi_1 e^{-\lambda \tau} - \lambda) [(\Xi_5 - \lambda)(\Xi_9 - \lambda) - \Xi_8 \Xi_6] - \Xi_2 [\Xi_4 (\Xi_9 - \lambda) e^{-\lambda \tau} - \Xi_7 \Xi_6 e^{-\lambda \tau}] \\ + \Xi_3 [\Xi_8 \Xi_4 e^{-\lambda \tau} - \Xi_7 (\Xi_5 - \lambda) e^{-\lambda \tau}] \quad (59)$$

$$\det(A - \lambda I) = (\Xi_1 e^{-\lambda \tau} - \lambda) [(\Xi_5 \Xi_9 - \Xi_8 \Xi_6) - (\Xi_5 + \Xi_9) \lambda + \lambda^2] - \Xi_2 (\Xi_4 \Xi_9 e^{-\lambda \tau} - \Xi_4 \lambda e^{-\lambda \tau} - \Xi_7 \Xi_6 e^{-\lambda \tau}) \\ + \Xi_3 (\Xi_8 \Xi_4 e^{-\lambda \tau} - \Xi_7 \Xi_5 e^{-\lambda \tau} + \Xi_7 \lambda e^{-\lambda \tau}) \quad (60)$$

$$\det(A - \lambda I) = [\Xi_5 \Xi_9 - \Xi_8 \Xi_6] \Xi_1 e^{-\lambda \tau} - (\Xi_5 + \Xi_9) \lambda \Xi_1 e^{-\lambda \tau} + \lambda^2 \Xi_1 e^{-\lambda \tau} - [\Xi_5 \Xi_9 - \Xi_8 \Xi_6] \lambda \\ + (\Xi_5 + \Xi_9) \lambda^2 - \lambda^3 - \Xi_2 \Xi_4 \Xi_9 e^{-\lambda \tau} + \Xi_2 \Xi_4 \lambda e^{-\lambda \tau} + \Xi_2 \Xi_7 \Xi_6 e^{-\lambda \tau} + \Xi_3 \Xi_8 \Xi_4 e^{-\lambda \tau} - \Xi_3 \Xi_7 \Xi_5 e^{-\lambda \tau} \\ + \Xi_3 \Xi_7 \lambda e^{-\lambda \tau} \quad (61)$$

$$\det(A - \lambda I) = -[\Xi_5 \Xi_9 - \Xi_8 \Xi_6] \lambda + (\Xi_5 + \Xi_9) \lambda^2 - \lambda^3 \\ + \{ [\Xi_5 \Xi_9 - \Xi_8 \Xi_6] \Xi_1 - \Xi_2 \Xi_4 \Xi_9 + \Xi_2 \Xi_7 \Xi_6 + \Xi_3 \Xi_8 \Xi_4 - \Xi_3 \Xi_7 \Xi_5 \\ + (\Xi_2 \Xi_4 - (\Xi_5 + \Xi_9) \Xi_1 + \Xi_3 \Xi_7) \lambda + \lambda^2 \Xi_1 \} e^{-\lambda \tau} \quad (62)$$

We define for simplicity global parameters: $\Omega_1 = -[\Xi_5 \Xi_9 - \Xi_8 \Xi_6]$; $\Omega_2 = \Xi_5 + \Xi_9$; $\Omega_4 = \Xi_2 \Xi_4 - (\Xi_5 + \Xi_9) \Xi_1 + \Xi_3 \Xi_7$,
 $\Omega_3 = [\Xi_5 \Xi_9 - \Xi_8 \Xi_6] \Xi_1 - \Xi_2 \Xi_4 \Xi_9 + \Xi_2 \Xi_7 \Xi_6 + \Xi_3 \Xi_8 \Xi_4 - \Xi_3 \Xi_7 \Xi_5$.

$$\det(A - \lambda I) = \Omega_1 \lambda + \Omega_2 \lambda^2 - \lambda^3 + [\Omega_3 + \Omega_4 \lambda + \Xi_1 \lambda^2] e^{-\lambda \tau}; \det(A - \lambda I) = 0 \quad (63)$$

The general characteristics equation is $D(\lambda, \tau) = \Omega_1 \lambda + \Omega_2 \lambda^2 - \lambda^3 + [\Omega_3 + \Omega_4 \lambda + \Xi_1 \lambda^2] e^{-\lambda \tau}$. We study the occurrence of any possible stability switching resulting from the increase of value of the time delay τ for the general characteristics equation $D(\lambda, \tau)$ [12][13]. The characteristics equation is $D(\lambda, \tau) = P_n(\lambda, \tau) + Q_m(\lambda, \tau) e^{-\lambda \tau}$; the expression for $P_n(\lambda, \tau)$: $P_n(\lambda, \tau) = \sum_{k=0}^n \lambda^k p_k(\tau)$. The expression for $Q_m(\lambda, \tau)$: $Q_m(\lambda, \tau) = \sum_{k=0}^m \lambda^k q_k(\tau)$. The characteristics equation is $D(\lambda, \tau) = P_n(\lambda, \tau) + Q_m(\lambda, \tau) e^{-\lambda \tau}$ where $n=3$, $m=2$ and $n > m$. The expression for $P_n(\lambda, \tau)$: $P_n(\lambda, \tau) = \sum_{k=0}^n \lambda^k p_k(\tau)$; $p_0(\tau) = 0$; $p_1(\tau) = \Omega_1$, $p_2(\tau) = \Omega_2$; $p_3(\tau) = -1$. The expression for $Q_m(\lambda, \tau)$: $Q_m(\lambda, \tau) = \sum_{k=0}^{m=2} \lambda^k q_k(\tau)$; $q_0(\tau) = \Omega_3$; $q_1(\tau) = \Omega_4$, $q_2(\tau) = \Xi_1$. The homogeneous system for C_{ave} , $C_{ave}^{cathode}$, C_{ave}^{anode} leads to a characteristic equation for the eigenvalue λ . (1) If $\lambda = j\omega$, $\omega \in \mathbb{R}$ then $P(j\omega) + Q(j\omega) \neq 0$, (2) $\left| \frac{Q(\lambda)}{P(\lambda)} \right|$ is bounded for $|\lambda| \rightarrow \infty$, $\text{Re } \lambda \geq 0$ and no roots bifurcation from ∞ , (3) $F(\omega) = |P(j\omega)|^2 - |Q(j\omega)|^2$ has a finite number of zeros. Indeed, this is a polynomial in ω , (4) Each $\omega(q_i, q_k)$ of

$F(\omega) = 0$ is continuous and differentiable with respect to q_i, q_k .

$$P_n(\lambda = j\omega, \tau) = \sum_{k=0}^{n=3} \lambda^k p_k(\tau) = -\Omega_2 \omega^2 + j(\omega^3 + \Omega_1 \omega); Q_m(\lambda = j\omega, \tau) = \sum_{k=0}^{m=2} \lambda^k q_k(\tau) = \Omega_3 - \Xi_1 \omega^2 + j\Omega_4 \omega \quad (64)$$

$$F(\omega) = |P(j\omega)|^2 - |Q(j\omega)|^2 = (-\Omega_2 \omega^2)^2 + (\omega^3 + \Omega_1 \omega)^2 - [(\Omega_3 - \Xi_1 \omega^2)^2 + (\Omega_4 \omega)^2] = \omega^6 + (2\Omega_1 + \Omega_2^2) \omega^4 + \Omega_1^2 \omega^2 - [\Omega_3^2 - 2\Omega_3 \Xi_1 \omega^2 + \Xi_1^2 \omega^4 + \Omega_4^2 \omega^2] \quad (65)$$

$$F(\omega) = |P(j\omega)|^2 - |Q(j\omega)|^2 = \omega^6 + (2\Omega_1 + \Omega_2^2) \omega^4 + \Omega_1^2 \omega^2 - \Omega_3^2 + 2\Omega_3 \Xi_1 \omega^2 - \Xi_1^2 \omega^4 - \Omega_4^2 \omega^2 \quad (66)$$

$$F(\omega) = |P(j\omega)|^2 - |Q(j\omega)|^2 = \omega^6 + [(2\Omega_1 + \Omega_2^2) - \Xi_1^2] \omega^4 + [\Omega_1^2 + 2\Omega_3 \Xi_1 - \Omega_4^2] \omega^2 - \Omega_3^2 \quad (67)$$

We define $\Phi_0 = -\Omega_3^2$; $\Phi_2 = \Omega_1^2 + 2\Omega_3 \Xi_1 - \Omega_4^2$; $\Phi_4 = (2\Omega_1 + \Omega_2^2) - \Xi_1^2$; $\Phi_6 = 1$. $F(\omega) = 0$ implies $\sum_{k=0}^3 \Phi_{2k} \omega^{2k} = 0$

. Its roots are given by solving the polynomial ($\sum_{k=0}^3 \Phi_{2k} \omega^{2k} = 0$). $P_R(j\omega, \tau) = -\Omega_2 \omega^2$, $P_I(j\omega, \tau) = \omega^3 + \Omega_1 \omega$, $Q_R(j\omega, \tau) = \Omega_3 - \Xi_1 \omega^2$, $Q_I(j\omega, \tau) = \Omega_4 \omega$.

$$\sin \theta(\tau) = \sin \theta(\omega\tau) = g(\omega); \cos \theta(\tau) = \cos \theta(\omega\tau) = h(\omega) \quad (68)$$

$$\sin \theta(\tau) = \frac{-P_R(j\omega, \tau)Q_I(j\omega, \tau) + P_I(j\omega, \tau)Q_R(j\omega, \tau)}{|Q(j\omega, \tau)|^2} \quad (69)$$

$$\cos \theta(\tau) = -\frac{[P_R(j\omega, \tau)Q_R(j\omega, \tau) + P_I(j\omega, \tau)Q_I(j\omega, \tau)]}{|Q(j\omega, \tau)|^2} \quad (70)$$

The above $\sin \theta(\tau)$ and $\cos \theta(\tau)$ expressions are jointly with $F(\omega) = 0 \Rightarrow \sum_{k=0}^3 \Phi_{2k} \omega^{2k} = 0$. Eigenvalue $\lambda = 0$ is not a root of the characteristic equation. Furthermore, $P(\lambda)$, $Q(\lambda)$ are analytic functions as the coefficients in P and Q are real. We have $\overline{P(-j\omega)} = P(j\omega)$ and $\overline{Q(-j\omega)} = Q(j\omega)$ thus $\lambda = j\omega$; $\omega > 0$ may be an eigenvalue of the characteristic equation [14][15]. The analysis consists of identifying the roots of the characteristic equation situated on the imaginary axis of the complex λ plane, by increasing the parameters $\varepsilon_s, L_s, \varepsilon_p, L_p, \varepsilon_n, L_n, J_p, J_n, \dots$, $\text{Re } \lambda$ may, at the crossing, change its sign from (-) to (+), that is from a stable focus E^* to an unstable, or vice versa. This feature may be further assessed by examining the sign of the partial derivatives with respect to $\varepsilon_s, L_s, \varepsilon_p, L_p, \varepsilon_n, L_n, J_p, J_n$ parameters. Upon separating into real and imaginary parts, with $P = P_R + jP_I$; $Q = Q_R + jQ_I$ and $P_\omega = P_{R\omega} + jP_{I\omega}$; $Q_\omega = Q_{R\omega} + jQ_{I\omega}$; $P^2 = P_R^2 + P_I^2$.

When (X) can be any Lithium-ion batteries solid-phase electrolyte concentrations system parameters $\varepsilon_s, L_s, \varepsilon_p, L_p, \varepsilon_n, L_n, J_p, J_n, \dots$ and time delay parameter τ .

$$F_\omega = 2[(P_{R\omega}P_R + P_{I\omega}P_I) - (Q_{R\omega}Q_R + Q_{I\omega}Q_I)]; F_x = 2[(P_{Rx}P_R + P_{Ix}P_I) - (Q_{Rx}Q_R + Q_{Ix}Q_I)]; \omega_x = -\frac{F_x}{F_\omega} \quad (71)$$

We choose our specific parameter as time delay $x = \tau$ then $P_R = -\Omega_2 \omega^2$; $P_I = \omega^3 + \Omega_1 \omega$; $Q_R = \Omega_3 - \Xi_1 \omega^2$, $Q_I = \Omega_4 \omega$. The related derivatives are $P_{R\omega} = -2\Omega_2 \omega$; $P_{I\omega} = 3\omega^2 + \Omega_1$; $Q_{R\omega} = -2\Xi_1 \omega$, $Q_{I\omega} = \Omega_4$.

$$F_\tau = 2[(P_{R\tau}P_R + P_{I\tau}P_I) - (Q_{R\tau}Q_R + Q_{I\tau}Q_I)] = 0; V = (P_RP_{I\tau} - P_IP_{R\tau}) - (Q_RQ_{I\tau} - Q_IQ_{R\tau}) = 0 \quad (72)$$

$$U = (P_RP_{I\omega} - P_IP_{R\omega}) - (Q_RQ_{I\omega} - Q_IQ_{R\omega}) \quad (73)$$

$$U = (-\Omega_2 \omega^2 [3\omega^2 + \Omega_1] + 2\Omega_2 \omega^2 [\omega^2 + \Omega_1]) - (\Omega_4 [\Omega_3 - \Xi_1 \omega^2] + 2\Omega_4 \Xi_1 \omega^2) \quad (74)$$

$$U = -\Omega_2 \omega^2 [3\omega^2 + \Omega_1] + 2\Omega_2 \omega^2 [\omega^2 + \Omega_1] - \Omega_4 [\Omega_3 - \Xi_1 \omega^2] - 2\Omega_4 \Xi_1 \omega^2 \quad (75)$$

By differentiating $F(\omega, \tau)$ with respect to τ , we get $F_\omega \frac{\partial \omega}{\partial \tau} + F_\tau = 0$; $\tau \in I \Rightarrow \frac{\partial \omega}{\partial \tau} = -\frac{F_\tau}{F_\omega}$.

$$\Lambda^{-1}(\tau) = \left(\frac{\partial \operatorname{Re} \lambda}{\partial \tau} \right)_{\lambda=j\omega}; \quad \frac{\partial \omega}{\partial \tau} = \omega_\tau = -\frac{F_\tau}{F_\omega}; \quad \Lambda^{-1}(\tau) = \operatorname{Re} \left\{ \frac{-2[U + \tau |P|^2] + jF_\omega}{F_\tau + j2[V + \omega |P|^2]} \right\} \quad (76)$$

$$\operatorname{sign}\{\Lambda^{-1}(\tau)\} = \operatorname{sign}\{F_\omega\} \operatorname{sign}\left\{ \tau \frac{\partial \omega}{\partial \tau} + \omega + \frac{U \frac{\partial \omega}{\partial \tau} + V}{|P|^2} \right\} \quad (77)$$

Since $F_\tau = 0 \Rightarrow \frac{\partial \omega}{\partial \tau} = \omega_\tau = -\frac{F_\tau}{F_\omega} \Big|_{F_\omega=0} = 0$ and $V = 0$ then $\operatorname{sign}\{\Lambda^{-1}(\tau)\} = \operatorname{sign}\{F_\omega\} \operatorname{sign}\{\omega\}$. The following $\operatorname{sign}\{\Lambda^{-1}(\tau)\}$ examines the possibility of stability transitions (bifurcations) Lithium-ion batteries solid-phase electrolyte concentrations system, about the equilibrium point E^* as a result of a variation of delay τ . We identify the roots of our system characteristic equation situated on the imaginary axis of the complex

λ - plane whereby increasing the delay parameter τ , $\operatorname{Re} \lambda$ may at the crossing, change its sign from - to +, that is from a stable focus E^* to an unstable one, or vice versa. We get the expression for $F(\omega, \tau)$ Lithium-ion batteries solid-phase electrolyte concentrations system parameter values. We define those ω, τ values which fulfill $F(\omega, \tau) = 0$. We ignore negative, complex, and imaginary values of ω for specific τ values [16][17].

Lithium-ion batteries solid-phase diffusion Approximations, transient response, electrolyte concentrations characteristic equation $\tau_1=\tau_2=0, \tau_3=\tau$

We get the expressions for $\xi_1 = \xi_1(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau)$ and $\xi_3 = \xi_3(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau)$, then

$$\xi_3 = Y_1 C_{\text{ave}}^{\text{cathode}}(t) + Y_2 C_{\text{ave}}(t) + Y_3 C_{\text{ave}}^{\text{anode}}(t - \tau) \quad (78)$$

$$\xi_1 = Y_4 C_{\text{ave}}^{\text{cathode}}(t) + Y_5 C_{\text{ave}}(t) + Y_6 C_{\text{ave}}^{\text{anode}}(t - \tau) \quad (79)$$

The standard local stability analysis about any one of the equilibrium points of Lithium-ion batteries solid-phase electrolyte concentrations system consists in adding to coordinate $[C_{\text{ave}}, C_{\text{ave}}^{\text{cathode}}, C_{\text{ave}}^{\text{anode}}]$ arbitrarily small increments of exponential form $[c_{\text{ave}}, c_{\text{ave}}^{\text{cathode}}, c_{\text{ave}}^{\text{anode}}]e^{\lambda t}$ and retaining the first order term in $c_{\text{ave}}, c_{\text{ave}}^{\text{cathode}}, c_{\text{ave}}^{\text{anode}}$. The system of three homogeneous equations leads to a polynomial characteristic equation in the eigenvalue λ [18][19]. The Lithium-ion batteries solid-phase electrolyte concentrations system fixed values with arbitrarily small increments of exponential form $[c_{\text{ave}}, c_{\text{ave}}^{\text{cathode}}, c_{\text{ave}}^{\text{anode}}]e^{\lambda t}$ are $k=1$ (first fixed point), $k=2$ (second fixed point), $k=3$ (third fixed point), ...

$$C_{\text{ave}}(t) = [C_{\text{ave}}]^* + c_{\text{ave}} e^{\lambda t}; \quad C_{\text{ave}}^{\text{cathode}}(t) = [C_{\text{ave}}^{\text{cathode}}]^* + c_{\text{ave}}^{\text{cathode}} e^{\lambda t}; \quad C_{\text{ave}}^{\text{anode}}(t) = [C_{\text{ave}}^{\text{anode}}]^* + c_{\text{ave}}^{\text{anode}} e^{\lambda t} \quad (80)$$

$$C_{\text{ave}}(t - \tau) = [C_{\text{ave}}]^* + c_{\text{ave}} e^{\lambda(t-\tau)}, \quad C_{\text{ave}}^{\text{cathode}}(t - \tau) = [C_{\text{ave}}^{\text{cathode}}]^* + c_{\text{ave}}^{\text{cathode}} e^{\lambda(t-\tau)}, \quad C_{\text{ave}}^{\text{anode}}(t - \tau) = [C_{\text{ave}}^{\text{anode}}]^* + c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} \quad (81)$$

$$\xi_3 = Y_1([C_{\text{ave}}^{\text{cathode}}]^* + c_{\text{ave}}^{\text{cathode}} e^{\lambda t}) + Y_2([C_{\text{ave}}]^* + c_{\text{ave}} e^{\lambda t}) + Y_3([C_{\text{ave}}^{\text{anode}}]^* + c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}) \quad (82)$$

$$\begin{aligned} \xi_3(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau) &= Y_1[C_{\text{ave}}^{\text{cathode}}]^* + Y_2[C_{\text{ave}}]^* + Y_3[C_{\text{ave}}^{\text{anode}}]^* \\ &+ Y_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + Y_2 c_{\text{ave}} e^{\lambda t} + Y_3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} \end{aligned} \quad (83)$$

We define ξ_3^* fixed expression as $\xi_3^* = Y_1[C_{ave}^{cathode}]^* + Y_2[C_{ave}]^* + Y_3[C_{ave}^{anode}]^*$ then $\xi_3(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau)$ is

$$\xi_3(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau) = \xi_3^* + Y_1 c_{ave}^{cathode} e^{\lambda t} + Y_2 c_{ave} e^{\lambda t} + Y_3 c_{ave}^{anode} e^{\lambda(t-\tau)}.$$

$$\xi_1(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau) = Y_4([C_{ave}^{cathode}]^* + c_{ave}^{cathode} e^{\lambda t}) + Y_5([C_{ave}]^* + c_{ave} e^{\lambda t}) + Y_6([C_{ave}^{anode}]^* + c_{ave}^{anode} e^{\lambda(t-\tau)}) \quad (84)$$

$$\xi_1(t, \tau_1 = \tau, \tau_2 = 0, \tau_3 = 0) = Y_4[C_{ave}^{cathode}]^* + Y_5[C_{ave}]^* + Y_6[C_{ave}^{anode}]^* + Y_4 c_{ave}^{cathode} e^{\lambda t} + Y_5 c_{ave} e^{\lambda t} + Y_6 c_{ave}^{anode} e^{\lambda(t-\tau)} \quad (85)$$

We define ξ_1^* fixed expression as $\xi_1^* = Y_4[C_{ave}^{cathode}]^* + Y_5[C_{ave}]^* + Y_6[C_{ave}^{anode}]^*$ then $\xi_1(t, \tau_1 = 0, \tau_2 = \tau_3 = \tau)$

$$\text{is } \xi_1(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau) = \xi_1^* + Y_4 c_{ave}^{cathode} e^{\lambda t} + Y_5 c_{ave} e^{\lambda t} + Y_6 c_{ave}^{anode} e^{\lambda(t-\tau)}.$$

We choose the above expressions for our $C_{ave}^{cathode}(t), C_{ave}(t), C_{ave}^{anode}(t)$ as small displacement $c_{ave}, c_{ave}^{cathode}, c_{ave}^{anode}$ from the system fixed points at time $t = 0$ ($C_{ave} = C_{ave}(t), C_{ave}^{cathode} = C_{ave}^{cathode}(t), C_{ave}^{anode} = C_{ave}^{anode}(t)$).

$$\frac{dC_{ave}(t)}{dt} = c_{ave} \lambda e^{\lambda t}; \quad \frac{dC_{ave}^{cathode}(t)}{dt} = c_{ave}^{cathode} \lambda e^{\lambda t}; \quad \frac{dC_{ave}^{anode}(t)}{dt} = c_{ave}^{anode} \lambda e^{\lambda t} \quad (86)$$

$$\frac{dC_{ave}(t-\tau)}{dt} = c_{ave} \lambda e^{\lambda(t-\tau)}, \quad \frac{dC_{ave}^{cathode}(t-\tau)}{dt} = c_{ave}^{cathode} \lambda e^{\lambda(t-\tau)}, \quad \frac{dC_{ave}^{anode}(t-\tau)}{dt} = c_{ave}^{anode} \lambda e^{\lambda(t-\tau)} \quad (87)$$

Under our specific delay parameters value ($\tau_1 = 0, \tau_2 = 0, \tau_3 = \tau$), we get the following system delay differential equations (DDEs):

$$\frac{dC_{ave}}{dt} = \frac{-6\varepsilon_s^3(2C_{ave}(t) - \xi_1 - \xi_3)}{L_s^2}; \quad \frac{dC_{ave}^{cathode}}{dt} = \frac{3\varepsilon_p^3(\xi_1 - C_{ave}^{cathode}(t))}{L_p^2} + J_p \text{ and}$$

$$\frac{dC_{ave}^{anode}}{dt} = \frac{3\varepsilon_n^3(\xi_3 - C_{ave}^{anode}(t-\tau))(1 - L_p - L_s)}{L_n^3} + J_n \quad [20][21].$$

The above ξ_1, ξ_3 expressions are related to our specific case ($\tau_1 = 0, \tau_2 = 0, \tau_3 = \tau$), $\xi_1(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau)$ and $\xi_3(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau)$.

$$\begin{aligned} & -6\varepsilon_s^3\{2([C_{ave}]^* + c_{ave} e^{\lambda t}) - (\xi_1^* + Y_4 c_{ave}^{cathode} e^{\lambda t} + Y_5 c_{ave} e^{\lambda t} + Y_6 c_{ave}^{anode} e^{\lambda(t-\tau)}) \\ & - (\xi_3^* + Y_1 c_{ave}^{cathode} e^{\lambda t} + Y_2 c_{ave} e^{\lambda t} + Y_3 c_{ave}^{anode} e^{\lambda(t-\tau)})\} = \frac{-6\varepsilon_s^3(2C_{ave}(t) - \xi_1 - \xi_3)}{L_s^2} \Rightarrow c_{ave} \lambda e^{\lambda t} = \frac{-6\varepsilon_s^3\{2([C_{ave}]^* + c_{ave} e^{\lambda t}) - (\xi_1^* + Y_4 c_{ave}^{cathode} e^{\lambda t} + Y_5 c_{ave} e^{\lambda t} + Y_6 c_{ave}^{anode} e^{\lambda(t-\tau)}) - (\xi_3^* + Y_1 c_{ave}^{cathode} e^{\lambda t} + Y_2 c_{ave} e^{\lambda t} + Y_3 c_{ave}^{anode} e^{\lambda(t-\tau)})\}}{L_s^2} \end{aligned} \quad (88)$$

$$\begin{aligned} & -6\varepsilon_s^3\{2([C_{ave}]^* + c_{ave} e^{\lambda t}) - (\xi_1^* + Y_4 c_{ave}^{cathode} e^{\lambda t} + Y_5 c_{ave} e^{\lambda t} + Y_6 c_{ave}^{anode} e^{\lambda(t-\tau)}) \\ & - (\xi_3^* + Y_1 c_{ave}^{cathode} e^{\lambda t} + Y_2 c_{ave} e^{\lambda t} + Y_3 c_{ave}^{anode} e^{\lambda(t-\tau)})\} = -6\varepsilon_s^3 2[C_{ave}]^* - 6\varepsilon_s^3 2c_{ave} e^{\lambda t} \\ & + 6\varepsilon_s^3 \xi_1^* + 6\varepsilon_s^3 Y_4 c_{ave}^{cathode} e^{\lambda t} + 6\varepsilon_s^3 Y_5 c_{ave} e^{\lambda t} + 6\varepsilon_s^3 Y_6 c_{ave}^{anode} e^{\lambda(t-\tau)} \\ & + 6\varepsilon_s^3 \xi_3^* + 6\varepsilon_s^3 Y_1 c_{ave}^{cathode} e^{\lambda t} + 6\varepsilon_s^3 Y_2 c_{ave} e^{\lambda t} + 6\varepsilon_s^3 Y_3 c_{ave}^{anode} e^{\lambda(t-\tau)} \end{aligned} \quad (89)$$

$$\begin{aligned}
 & -6\varepsilon_s^3 \{2([C_{\text{ave}}]^* + c_{\text{ave}} e^{\lambda t}) - (\xi_1^* + \Upsilon_4 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_5 c_{\text{ave}} e^{\lambda t} + \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}) \\
 & - (\xi_3^* + \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_2 c_{\text{ave}} e^{\lambda t} + \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)})\} = -6(\varepsilon_s^3 2[C_{\text{ave}}]^* - \varepsilon_s^3 \xi_1^* - \varepsilon_s^3 \xi_3^*) \\
 & + 6\{\varepsilon_s^3 [-2 + \Upsilon_5 + \Upsilon_2] c_{\text{ave}} + \varepsilon_s^3 [\Upsilon_4 + \Upsilon_1] c_{\text{ave}}^{\text{cathode}} + \varepsilon_s^3 [\Upsilon_6 + \Upsilon_3] c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau}\} e^{\lambda t}
 \end{aligned} \quad (90)$$

We get the following first eigenvalue function:

$$c_{\text{ave}} \lambda e^{\lambda t} = \frac{-6(\varepsilon_s^3 2[C_{\text{ave}}]^* - \varepsilon_s^3 \xi_1^* - \varepsilon_s^3 \xi_3^*) + 6\{\varepsilon_s^3 [-2 + \Upsilon_5 + \Upsilon_2] c_{\text{ave}} + \varepsilon_s^3 [\Upsilon_4 + \Upsilon_1] c_{\text{ave}}^{\text{cathode}} + \varepsilon_s^3 [\Upsilon_6 + \Upsilon_3] c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau}\} e^{\lambda t}}{L_s^2} \quad (91)$$

$$c_{\text{ave}} \lambda e^{\lambda t} = \frac{-6\varepsilon_s^3 (2[C_{\text{ave}}]^* - \xi_1^* - \xi_3^*)}{L_s^2} + \frac{\varepsilon_s^3 [\Upsilon_4 + \Upsilon_1] c_{\text{ave}}^{\text{cathode}} + \varepsilon_s^3 [\Upsilon_6 + \Upsilon_3] c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau}}{L_s^2} e^{\lambda t} \quad (92)$$

Since $\frac{dC_{\text{ave}}}{dt} = \frac{-6\varepsilon_s^3 (2C_{\text{ave}}(t) - \xi_1 - \xi_3)}{L_s^2}$, at fixed point $(\frac{dC_{\text{ave}}}{dt} = 0; \lim_{\substack{t \rightarrow \infty \\ t \gg \tau}} C_{\text{ave}}^{\text{anode}}(t - \tau) = C_{\text{ave}}^{\text{anode}}(t))$ exists

$$\frac{-6\varepsilon_s^3 (2[C_{\text{ave}}]^* - \xi_1^* - \xi_3^*)}{L_s^2} = 0 \text{ and the first eigenvalue equation is}$$

$$c_{\text{ave}} \lambda = \frac{6\varepsilon_s^3 [-2 + \Upsilon_5 + \Upsilon_2]}{L_s^2} c_{\text{ave}} + \frac{6\varepsilon_s^3 [\Upsilon_4 + \Upsilon_1]}{L_s^2} c_{\text{ave}}^{\text{cathode}} + \frac{6\varepsilon_s^3 [\Upsilon_6 + \Upsilon_3] c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau}}{L_s^2} \quad (93)$$

$$\left(\frac{6\varepsilon_s^3 [-2 + \Upsilon_5 + \Upsilon_2]}{L_s^2} - \lambda \right) c_{\text{ave}} + \frac{6\varepsilon_s^3 [\Upsilon_4 + \Upsilon_1]}{L_s^2} c_{\text{ave}}^{\text{cathode}} + \frac{6\varepsilon_s^3 [\Upsilon_6 + \Upsilon_3] c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau}}{L_s^2} = 0 \quad (94)$$

We define for simplicity system global parameters: $\Xi_1 = \frac{6\varepsilon_s^3 [-2 + \Upsilon_5 + \Upsilon_2]}{L_s^2}$; $\Xi_2 = \frac{6\varepsilon_s^3 [\Upsilon_4 + \Upsilon_1]}{L_s^2}$

$\Xi_3 = \frac{6\varepsilon_s^3 [\Upsilon_6 + \Upsilon_3]}{L_s^2}$ then the first eigenvalue equation is $(\Xi_1 - \lambda) c_{\text{ave}} + \Xi_2 c_{\text{ave}}^{\text{cathode}} + \Xi_3 e^{-\lambda \tau} c_{\text{ave}}^{\text{anode}} = 0$.

$$\frac{dC_{\text{ave}}^{\text{cathode}}}{dt} = \frac{3\varepsilon_p^3 (\xi_1 - C_{\text{ave}}^{\text{cathode}}(t))}{L_p^2} + J_p \Rightarrow c_{\text{ave}}^{\text{cathode}} \lambda e^{\lambda t} = \frac{3\varepsilon_p^3 [(\xi_1^* + \Upsilon_4 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_5 c_{\text{ave}} e^{\lambda t} + \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}) - ([C_{\text{ave}}^{\text{cathode}}]^* + c_{\text{ave}}^{\text{cathode}} e^{\lambda t})]}{L_p^2} + J_p \quad (95)$$

$$\begin{aligned}
 & 3\varepsilon_p^3 [(\xi_1^* + \Upsilon_4 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_5 c_{\text{ave}} e^{\lambda t} + \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}) - ([C_{\text{ave}}^{\text{cathode}}]^* + c_{\text{ave}}^{\text{cathode}} e^{\lambda t})] = 3\varepsilon_p^3 \xi_1^* \\
 & + 3\varepsilon_p^3 \Upsilon_4 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_5 c_{\text{ave}} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} - 3\varepsilon_p^3 [C_{\text{ave}}^{\text{cathode}}]^* - 3\varepsilon_p^3 c_{\text{ave}}^{\text{cathode}} e^{\lambda t}
 \end{aligned} \quad (96)$$

$$\begin{aligned}
 & 3\varepsilon_p^3 \xi_1^* + 3\varepsilon_p^3 \Upsilon_4 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_5 c_{\text{ave}} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} - 3\varepsilon_p^3 [C_{\text{ave}}^{\text{cathode}}]^* - 3\varepsilon_p^3 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} \\
 & = 3\varepsilon_p^3 (\xi_1^* - [C_{\text{ave}}^{\text{cathode}}]^*) + 3\varepsilon_p^3 \Upsilon_5 c_{\text{ave}} e^{\lambda t} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}
 \end{aligned} \quad (97)$$

We get the following second eigenvalue function:

$$c_{\text{ave}}^{\text{cathode}} \lambda e^{\lambda t} = \frac{3\varepsilon_p^3 (\xi_1^* - [C_{\text{ave}}^{\text{cathode}}]^*) + 3\varepsilon_p^3 \Upsilon_5 c_{\text{ave}} e^{\lambda t} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}}{L_p^2} + J_p \quad (98)$$

$$c_{\text{ave}}^{\text{cathode}} \lambda e^{\lambda t} = \frac{3\varepsilon_p^3 (\xi_1^* - [C_{\text{ave}}^{\text{cathode}}]^*)}{L_p^2} + J_p + \frac{3\varepsilon_p^3 \Upsilon_5 c_{\text{ave}} e^{\lambda t} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + 3\varepsilon_p^3 \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}}{L_p^2} \quad (99)$$

Since $\frac{dC_{\text{ave}}^{\text{cathode}}}{dt} = \frac{3\varepsilon_p^3 (\xi_1^* - C_{\text{ave}}^{\text{cathode}}(t))}{L_p^2} + J_p$, at fixed point ($\frac{dC_{\text{ave}}^{\text{cathode}}}{dt} = 0$) exists

$$\frac{3\varepsilon_p^3 (\xi_1^* - [C_{\text{ave}}^{\text{cathode}}]^*)}{L_p^2} + J_p = 0 \text{ and the second eigenvalue equation is}$$

$$c_{\text{ave}}^{\text{cathode}} \lambda = \frac{3\varepsilon_p^3 \Upsilon_5 c_{\text{ave}} + 3\varepsilon_p^3 (\Upsilon_4 - 1) c_{\text{ave}}^{\text{cathode}} + 3\varepsilon_p^3 \Upsilon_6 c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau}}{L_p^2} \quad (100)$$

$$\frac{3\varepsilon_p^3 \Upsilon_5}{L_p^2} c_{\text{ave}} + \left(\frac{3\varepsilon_p^3 (\Upsilon_4 - 1)}{L_p^2} - \lambda \right) c_{\text{ave}}^{\text{cathode}} + \frac{3\varepsilon_p^3 \Upsilon_6}{L_p^2} c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau} = 0 \quad (101)$$

We define for simplicity system global parameters: $\Xi_4 = \frac{3\varepsilon_p^3 \Upsilon_5}{L_p^2}$; $\Xi_5 = \frac{3\varepsilon_p^3 (\Upsilon_4 - 1)}{L_p^2}$; $\Xi_6 = \frac{3\varepsilon_p^3 \Upsilon_6}{L_p^2}$ then the second eigenvalue equation is $\Xi_4 c_{\text{ave}} + (\Xi_5 - \lambda) c_{\text{ave}}^{\text{cathode}} + \Xi_6 c_{\text{ave}}^{\text{anode}} e^{-\lambda \tau} = 0$.

$$\frac{dC_{\text{ave}}^{\text{anode}}}{dt} = \frac{3\varepsilon_n^3 (\xi_3^* - C_{\text{ave}}^{\text{anode}}(t-\tau))(1-L_p-L_s)}{L_n^3} + J_n \Rightarrow c_{\text{ave}}^{\text{anode}} \lambda e^{\lambda t} = \frac{3\varepsilon_n^3 ([\xi_3^* + \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_2 c_{\text{ave}} e^{\lambda t} + \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}] - [[C_{\text{ave}}^{\text{anode}}]^* + c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}])(1-L_p-L_s)}{L_n^3} + J_n \quad (102)$$

$$\text{Since } \xi_3(t, \tau_1 = 0, \tau_2 = 0, \tau_3 = \tau) = \xi_3^* + \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_2 c_{\text{ave}} e^{\lambda t} + \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}$$

$$3\varepsilon_n^3 ([\xi_3^* + \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} + \Upsilon_2 c_{\text{ave}} e^{\lambda t} + \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}] - [[C_{\text{ave}}^{\text{anode}}]^* + c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)}])(1-L_p-L_s) \quad (103)$$

$$= 3\varepsilon_n^3 \xi_3^* (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda t} (1-L_p-L_s)$$

$$+ 3\varepsilon_n^3 \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} (1-L_p-L_s) - 3\varepsilon_n^3 [C_{\text{ave}}^{\text{anode}}]^* (1-L_p-L_s) - 3\varepsilon_n^3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} (1-L_p-L_s)$$

$$3\varepsilon_n^3 \xi_3^* (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda t} (1-L_p-L_s) \quad (104)$$

$$+ 3\varepsilon_n^3 \Upsilon_3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} (1-L_p-L_s) - 3\varepsilon_n^3 [C_{\text{ave}}^{\text{anode}}]^* (1-L_p-L_s) - 3\varepsilon_n^3 c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} (1-L_p-L_s)$$

$$= 3\varepsilon_n^3 (\xi_3^* - [C_{\text{ave}}^{\text{anode}}]^*) (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda t} (1-L_p-L_s)$$

$$+ 3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 [\Upsilon_3 - 1] c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} (1-L_p-L_s)$$

We get the following third eigenvalue function:

$$c_{\text{ave}}^{\text{anode}} \lambda e^{\lambda t} = \frac{3\varepsilon_n^3 (\xi_3^* - [C_{\text{ave}}^{\text{anode}}]^*) (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_2 c_{\text{ave}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 \Upsilon_1 c_{\text{ave}}^{\text{cathode}} e^{\lambda t} (1-L_p-L_s) + 3\varepsilon_n^3 [\Upsilon_3 - 1] c_{\text{ave}}^{\text{anode}} e^{\lambda(t-\tau)} (1-L_p-L_s)}{L_n^3} + J_n \quad (105)$$

$$c_{ave}^{anode} \lambda e^{\lambda t} = \frac{3\varepsilon_n^3 (\xi_3^* - [C_{ave}^{anode}]^*) (1 - L_p - L_s)}{L_n^3} + J_n + \frac{3\varepsilon_n^3 \Upsilon_2 c_{ave} e^{\lambda t} (1 - L_p - L_s)}{L_n^3} \\ + \frac{3\varepsilon_n^3 \Upsilon_1 c_{ave}^{cathode} e^{\lambda t} (1 - L_p - L_s)}{L_n^3} + \frac{3\varepsilon_n^3 [\Upsilon_3 - 1] c_{ave}^{anode} e^{\lambda(t-\tau)} (1 - L_p - L_s)}{L_n^3} \quad (106)$$

Since $\frac{dC_{ave}^{anode}}{dt} = \frac{3\varepsilon_n^3 (\xi_3 - C_{ave}^{anode}(t)) (1 - L_p - L_s)}{L_n^3} + J_n$, at fixed point ($\frac{dC_{ave}^{anode}}{dt} = 0$) exists $\frac{3\varepsilon_n^3 (\xi_3^* - [C_{ave}^{anode}]^*) (1 - L_p - L_s)}{L_n^3} = 0$ and the third eigenvalue equation is

$$c_{ave}^{anode} \lambda = \frac{3\varepsilon_n^3 \Upsilon_2 (1 - L_p - L_s)}{L_n^3} c_{ave} + \frac{3\varepsilon_n^3 \Upsilon_1 (1 - L_p - L_s)}{L_n^3} c_{ave}^{cathode} + \frac{3\varepsilon_n^3 [\Upsilon_3 - 1] (1 - L_p - L_s)}{L_n^3} e^{-\lambda \tau} c_{ave}^{anode} \quad (107)$$

We define for simplicity system global parameters: $\Xi_7 = \frac{3\varepsilon_n^3 \Upsilon_2 (1 - L_p - L_s)}{L_n^3}$; $\Xi_8 = \frac{3\varepsilon_n^3 \Upsilon_1 (1 - L_p - L_s)}{L_n^3}$

$\Xi_9 = \frac{3\varepsilon_n^3 [\Upsilon_3 - 1] (1 - L_p - L_s)}{L_n^3}$ then the third eigenvalue equation is $c_{ave}^{anode} \lambda = \Xi_7 c_{ave} + \Xi_8 c_{ave}^{cathode} + \Xi_9 e^{-\lambda \tau} c_{ave}^{anode}$.

We can summary our system three eigenvalues equation [22][23] with delay parameter (τ):

First eigenvalue equation: $(\Xi_1 - \lambda) c_{ave} + \Xi_2 c_{ave}^{cathode} + \Xi_3 e^{-\lambda \tau} c_{ave}^{anode} = 0$

Second eigenvalue equation: $\Xi_4 c_{ave} + (\Xi_5 - \lambda) c_{ave}^{cathode} + \Xi_6 c_{ave}^{anode} e^{-\lambda \tau} = 0$

Third eigenvalue equation: $c_{ave}^{anode} \lambda = \Xi_7 c_{ave} + \Xi_8 c_{ave}^{cathode} + \Xi_9 e^{-\lambda \tau} c_{ave}^{anode}$

The eigenvalue matrix we get is
$$\begin{pmatrix} (\Xi_1 - \lambda) & \Xi_2 & \Xi_3 e^{-\lambda \tau} \\ \Xi_4 & (\Xi_5 - \lambda) & \Xi_6 e^{-\lambda \tau} \\ \Xi_7 & \Xi_8 & (\Xi_9 - \lambda) e^{-\lambda \tau} \end{pmatrix} \begin{pmatrix} c_{ave} \\ c_{ave}^{cathode} \\ c_{ave}^{anode} \end{pmatrix} = 0$$

$$A - \lambda I = \begin{pmatrix} (\Xi_1 - \lambda) & \Xi_2 & \Xi_3 e^{-\lambda \tau} \\ \Xi_4 & (\Xi_5 - \lambda) & \Xi_6 e^{-\lambda \tau} \\ \Xi_7 & \Xi_8 & (\Xi_9 - \lambda) e^{-\lambda \tau} \end{pmatrix}; \det(A - \lambda I) = 0 \quad (108)$$

$$\det(A - \lambda I) = (\Xi_1 - \lambda) \det \begin{pmatrix} (\Xi_5 - \lambda) & \Xi_6 e^{-\lambda \tau} \\ \Xi_8 & (\Xi_9 - \lambda) e^{-\lambda \tau} \end{pmatrix} - \Xi_2 \det \begin{pmatrix} \Xi_4 & \Xi_6 e^{-\lambda \tau} \\ \Xi_7 & (\Xi_9 - \lambda) e^{-\lambda \tau} \end{pmatrix} \\ + \Xi_3 e^{-\lambda \tau} \det \begin{pmatrix} \Xi_4 & (\Xi_5 - \lambda) \\ \Xi_7 & \Xi_8 \end{pmatrix} \quad (109)$$

$$\det(A - \lambda I) = (\Xi_1 - \lambda) [(\Xi_5 - \lambda)(\Xi_9 - \lambda) e^{-\lambda \tau} - \Xi_8 \Xi_6 e^{-\lambda \tau}] - \Xi_2 [\Xi_4 (\Xi_9 - \lambda) e^{-\lambda \tau} - \Xi_7 \Xi_6 e^{-\lambda \tau}] \\ + \Xi_3 e^{-\lambda \tau} [\Xi_4 \Xi_8 - \Xi_7 (\Xi_5 - \lambda)] \quad (110)$$

$$\det(A - \lambda I) = (\Xi_1 - \lambda)[(\Xi_5 \Xi_9 - \lambda(\Xi_5 + \Xi_9) + \lambda^2)e^{-\lambda\tau} - \Xi_8 \Xi_6 e^{-\lambda\tau}] - \Xi_2[\Xi_4 \Xi_9 e^{-\lambda\tau} - \Xi_4 \lambda e^{-\lambda\tau} - \Xi_7 \Xi_6 e^{-\lambda\tau}] + \Xi_3 e^{-\lambda\tau}[\Xi_4 \Xi_8 - \Xi_5 \Xi_7 + \Xi_7 \lambda] \quad (111)$$

$$\det(A - \lambda I) = (\Xi_1 \Xi_5 \Xi_9 - \lambda \Xi_1 (\Xi_5 + \Xi_9) + \Xi_1 \lambda^2)e^{-\lambda\tau} - \Xi_1 \Xi_8 \Xi_6 e^{-\lambda\tau} - \lambda(\Xi_5 \Xi_9 - \lambda(\Xi_5 + \Xi_9) + \lambda^2)e^{-\lambda\tau} + \Xi_8 \Xi_6 \lambda e^{-\lambda\tau} - \Xi_2 \Xi_4 \Xi_9 e^{-\lambda\tau} + \Xi_2 \Xi_4 \lambda e^{-\lambda\tau} + \Xi_2 \Xi_7 \Xi_6 e^{-\lambda\tau} + \Xi_4 \Xi_8 \Xi_3 e^{-\lambda\tau} - \Xi_5 \Xi_7 \Xi_3 e^{-\lambda\tau} + \Xi_7 \Xi_3 \lambda e^{-\lambda\tau} \quad (112)$$

$$\det(A - \lambda I) = \Xi_1 \Xi_5 \Xi_9 e^{-\lambda\tau} - \Xi_1 (\Xi_5 + \Xi_9) \lambda e^{-\lambda\tau} + \Xi_1 \lambda^2 e^{-\lambda\tau} - \Xi_1 \Xi_8 \Xi_6 e^{-\lambda\tau} - \Xi_5 \Xi_9 \lambda e^{-\lambda\tau} + (\Xi_5 + \Xi_9) \lambda^2 e^{-\lambda\tau} - \lambda^3 e^{-\lambda\tau} + \Xi_8 \Xi_6 \lambda e^{-\lambda\tau} - \Xi_2 \Xi_4 \Xi_9 e^{-\lambda\tau} + \Xi_2 \Xi_4 \lambda e^{-\lambda\tau} + \Xi_2 \Xi_7 \Xi_6 e^{-\lambda\tau} + \Xi_4 \Xi_8 \Xi_3 e^{-\lambda\tau} - \Xi_5 \Xi_7 \Xi_3 e^{-\lambda\tau} + \Xi_7 \Xi_3 \lambda e^{-\lambda\tau} \quad (113)$$

$$\det(A - \lambda I) = [\Xi_1 \Xi_5 \Xi_9 - \Xi_2 \Xi_4 \Xi_9 - \Xi_1 \Xi_8 \Xi_6 + \Xi_2 \Xi_7 \Xi_6 + \Xi_4 \Xi_8 \Xi_3 - \Xi_5 \Xi_7 \Xi_3 + (\Xi_8 \Xi_6 + \Xi_2 \Xi_4 + \Xi_7 \Xi_3 - \Xi_5 \Xi_9 - \Xi_1 (\Xi_5 + \Xi_9))\lambda + (\Xi_5 + \Xi_9 + \Xi_1)\lambda^2 - \lambda^3]e^{-\lambda\tau} \quad (114)$$

We define for simplicity global parameters: $\Omega_0 = \Xi_1 \Xi_5 \Xi_9 - \Xi_2 \Xi_4 \Xi_9 - \Xi_1 \Xi_8 \Xi_6 + \Xi_2 \Xi_7 \Xi_6 + \Xi_4 \Xi_8 \Xi_3 - \Xi_5 \Xi_7 \Xi_3$,

$$\Omega_1 = \Xi_8 \Xi_6 + \Xi_2 \Xi_4 + \Xi_7 \Xi_3 - \Xi_5 \Xi_9 - \Xi_1 (\Xi_5 + \Xi_9); \quad \Omega_2 = \Xi_5 + \Xi_9 + \Xi_1; \quad \Omega_3 = -1.$$

Then we get $\det(A - \lambda I) = [\sum_{k=0}^3 \Omega_k \lambda^k] e^{-\lambda\tau}$; $\det(A - \lambda I) = 0$, the general characteristics equation is

$$D(\lambda, \tau) = [\sum_{k=0}^{m=3} \Omega_k \lambda^k] e^{-\lambda\tau}; \quad Q_m(\lambda, \tau) = \sum_{k=0}^{m=3} \Omega_k \lambda^k. \text{ We study the occurrence of any possible stability switching resulting}$$

from the increase of value of the time delay τ for the general characteristics equation $D(\lambda, \tau)$. The characteristics equation

is $D(\lambda, \tau) = P_n(\lambda, \tau) + Q_m(\lambda, \tau)e^{-\lambda\tau}$; the expression for $P_n(\lambda, \tau)$ is equal to zero: $P_n(\lambda, \tau) = 0$. The expression for $Q_m(\lambda, \tau)$

: $Q_m(\lambda, \tau) = \sum_{k=0}^m \lambda^k q_k(\tau)$. The characteristics equation is $D(\lambda, \tau) = P_n(\lambda, \tau) + Q_m(\lambda, \tau)e^{-\lambda\tau}$ where $P_n(\lambda, \tau) = 0$, $m = 3$ then

$\Omega_k(\lambda, \tau) = q_k(\tau) \forall k = 0, 1, 2, 3$. We get characteristics equation is $D(\lambda, \tau) = Q_m(\lambda, \tau)e^{-\lambda\tau}$. The $P_n(\lambda, \tau) = 0$ then the

characteristic equation is reduced to $Q_m(\lambda, \tau)e^{-\lambda\tau} = 0$. Since $e^{-\lambda\tau} \neq 0$ for any finite eigenvalue λ therefore the characteristic

equation can only be satisfied if $Q_m(\lambda, \tau) = 0$. Thus, when $P_n(\lambda, \tau) = 0$ the delay term (τ) not determine the characteristic roots. The roots are simply the zeros of $Q_m(\lambda, \tau)$ [24][25]. For any finite complex value of λ , the exponential term $e^{-\lambda\tau}$

can never equal to zero ($e^{-\lambda\tau} \neq 0$). Therefore, the roots of the system are entirely determined by the algebraic equation

$Q_m(\lambda)$. We can analyse stability using standard polynomial methods (like the Routh-Hurwitz criterion) on $Q_m(\lambda) = 0$

. To analyse the stability of a third-degree characteristic polynomial $Q_m(\lambda, \tau) = \sum_{k=0}^{m=3} \lambda^k q_k(\tau) = \sum_{k=0}^{m=3} \Omega_k \lambda^k$, we use the Routh-Hurwitz criterion. A system is stable only if all roots

(λ) have negative real parts. The third-degree characteristic polynomial is $Q(\lambda) = \sum_{k=0}^{m=3} \Omega_k \lambda^k$,

$Q(\lambda) = \sum_{k=0}^{m=3} \Omega_k \lambda^k = \Omega_0 + \Omega_1 \lambda + \Omega_2 \lambda^2 + \Omega_3 \lambda^3$, we assume the leading coefficient $\Omega_3 > 0$ (since $\Omega_3 = -1$, we multiply the

characteristic polynomial by (-1) and get the new global parameters: $\Omega_0 = -\Xi_1\Xi_5\Xi_9 + \Xi_2\Xi_4\Xi_9 + \Xi_1\Xi_8\Xi_6 - \Xi_2\Xi_7\Xi_6 - \Xi_4\Xi_8\Xi_3 + \Xi_5\Xi_7\Xi_3$,

$\Omega_1 = -\Xi_8\Xi_6 - \Xi_2\Xi_4 - \Xi_7\Xi_3 + \Xi_5\Xi_9 + \Xi_1(\Xi_5 + \Xi_9)$; $\Omega_2 = -(\Xi_5 + \Xi_9 + \Xi_1)$; $\Omega_3 = 1$. Now leading coefficient ($\Omega_3 > 0$), all other coefficients must be strictly positive: $\Omega_2 > 0$, $\Omega_1 > 0$, and $\Omega_0 > 0$ (necessary conditions). The sufficient condition (stable system) is $\Omega_2\Omega_1 > \Omega_3\Omega_0$ [26][27][28]. If all conditions are fully met the system is stable. If any condition fails or a coefficient is negative ($\Omega_k < 0 \forall k \in \{0, 1, 2, 3\}$) then the system is unstable. We get marginally stable system if $\Omega_2\Omega_1 = \Omega_3\Omega_0$ while all coefficients are positive. A marginally stable system is defined as a dynamic system that neither grows infinitely nor settles down to zero when given a finite push or input. Instead, the system produces continuous, un-damped oscillations or a permanent, bounded offset that lasts forever. If we want to inspect the system stability from eigenvalues (λ 's) then an unstable eigenvalue will have a positive or zero real part and a stable eigenvalue will have a negative real part. We inspect if any of the coefficients are negative or zero. If so, there is at least one value with a positive or zero real part which refers to an unstable node [29][30][31].

Discussion

We know when we use Lithium – ion batteries that there is a risk of overheating, catching fire, or exploding. It can be happened if the Li – ion batteries are damaged, overcharged, or are resided in high heat environment. Also, the Li – ion batteries slowly lose capacity over the years. We define the battery degradation factor when they exposure to high heat or keeping the battery at 100% which shortens their overall lifespan. The Li – ion batteries are steel popular in many applications due to the long lifespan, exceptional run time and fast charging ability of high energy density Lit – ion batteries. Nonlinearity is a critical behaviour when we inspect Lit – ion batteries deeply and additionally parasitic behaviour are involved in the dynamic of the battery. The key Lit – ion components include positive electrode, separator, and negative electrode [32][33]. All Lit – ion batteries components are characterized by plenty of parameters such as thickness, porosity, tortuosity, particle radius, solid diffusion coefficient, mass loading. The key parameters of Lit – ion separator are thickness, porosity, mean pore size, puncture strength, wettability, thermal shut down and stability. The Lit – ion battery is also evaluated by state – of charge (SOC) and state – of – health (SOH) estimation. When we know the battery real – time state and parameters info then inspection of the battery at limits without compromising durability is possible. The Lit – ion electrochemical influences the SOC and SOH estimation. The electrochemical process at Lit – ion battery describes the situation where lithium ions shuttle back and forth between a positive cathode and a negative anode. It is done via an ion conducting electrolyte and generate electricity through reversible reduction – oxidation reactions. The active materials which are selected for lithium-ion battery (LIB) need to have good electrical conductivity, ionic conductivity and cycling stability. Additionally, they need to have reversible capacity and low cost. We need to use non – aqueous electrolyte, as lithium has an interaction with water molecules which leads to safety issue. Typically, lithium – salts (LiPF₆) dissolved in organic liquid. Typical organic liquids are ethylene carbonate (EC), dimethyl carbonate (DMC) or diethyl carbonate (DEC) which are used as electrolytes [34][35].

The current Lit – ion battery DDEs model is restricted to electrolyte concentration dynamics only. We define lithium – ion cell battery Hopf bifurcation induced self – sustained oscillations as a representation of nonlinear dynamic behaviour. The nonlinear dynamic behaviour happened when the stable operation state makes a transition into voltage and temperature periodic fluctuations. The phenomenon actually happened under high – stress operating thresholds. The high – stress operating thresholds can be thermal runaway precursors or an extreme current rate. It is governed by structural time delay parameters and negative differential resistance (NDR) which it is within the cell chemistry. Lit – ion battery standard operation behaves as a damping system where possible perturbations are decayed back to an equilibrium steady state (stable fixed point). If we change the system control parameters such as current density, internal degradation or ambient temperature and their values are beyond specific threshold values then can happened a nonlinear system phenomenon. The nonlinear system phenomenon is related to a pair of complex conjugate eigenvalues of the battery governing electro – thermal state equations which cross the imaginary axis into the right – half plane. The crossing phenomenon is defined as Hopf bifurcation which stable operating point moves to unstable state and a periodic limit cycle emerges. In that case the Lit – ion battery state is no longer stabilizes and instead enters to continuous state, it is self – sustained oscillations.

Conclusion

A Lithium – ion (Li – ion) battery is used as a rechargeable energy storage device which is uses lithium ions. The battery lithium ions are moved between a cathode and an anode. The battery anode (negative) is made of graphite and it releases or holds lithium ions. The battery electrolyte is made from liquid or gel which lets ions to move back and forth. The battery

separator is a thin porous film that blocks electrons from causing internal shorts while letting ions pass. We get from Li – ion battery high energy density (more power in less space), low self – discharge (retains charge when not in use), and no memory effect since there is no need to be fully emptied before recharging. Battery electrolyte concentration is the key parameter when we want to define the battery performance [36][37]. The pore wall flux is a very important parameter value regarding the battery negative and positive electrodes. The Li – ion battery pore wall flux ($\dot{J}_{Li}$ or J) is defined as the volumetric transfer rate of lithium ions across the solid – electrolyte interface per unit area. It is the key source term regarding the porous electrode mass – conservation equations. We have key variables when defining Li – ion battery. The key variables are C_{ave} , $C_{ave}^{cathode}$, and C_{ave}^{anode} which describe the electrolyte concentration which are volume averaged over the respective region. The Li – ion battery rate equations (DDEs) give the battery transient response dynamic and integrate the parasitic effects of the battery dynamic. The parasitic effects are described by delay parameters in time τ_1 , τ_2 , and τ_3 . We discuss the stability of our system for the cases of shifting in time the C_{ave} variable ($C_{ave}(t) \rightarrow C_{ave}(t - \tau)$), $C_{ave}^{cathode}$ variable ($C_{ave}^{cathode}(t) \rightarrow C_{ave}^{cathode}(t - \tau)$), and C_{ave}^{anode} variable ($C_{ave}^{anode}(t) \rightarrow C_{ave}^{anode}(t - \tau)$). The time delay parameter (τ) is varied and influence the system dynamic and stability. The Li – ion rate equations (DDEs) include many battery parameters which influence also on system dynamic and stability. The Li – ion battery parameters describe the solid phase diffusion approximations [38][39]. The Li – ion battery rate equations (DDEs with C_{ave} , $C_{ave}^{cathode}$, and C_{ave}^{anode} variables) with delay parameters (τ_1 , τ_2 , and τ_3) are for different cases establish the system stability by analysing how past states influence current changes. The Li – ion battery system stability is dependent on transcendental characteristic equation with infinite roots or through energy – like functionals. If all characteristic roots stay in the left half of the complex plane the delayed system is stable. We get characteristic matrix determinant set to zero, containing exponential term ($e^{-\lambda\tau}$). The delay differential equations (DDEs) have a finite number of characteristic roots (λ) and is asymptotically stable only if every root has a negative real part ($\text{Re}(\lambda) < 0$). If even one root crosses into the positive right – half plane ($\text{Re}(\lambda) > 0$) then the system becomes unstable. We use Beretta and Kuang stability switch criteria which is used to find changes in stability (stability switches) and Hopf bifurcations in delay differential equations (DDEs). The system parameters are dependent on the delay parameter (τ). The roots of transcendental characteristic equations are linked to the frequency and size of the time delay. We need to track the imaginary roots for determining when purely imaginary eigenvalues cross the imaginary axis as a time delay parameter (τ) increases. The crossing direction uses the sign of the derivative of the roots with respect to delay parameter (τ). All of that for proving whether a system switches from stable to unstable, or vice versa. Zero roots ($\lambda = 0$) are never a characteristic root of the system and also roots cannot bifurcate from infinity as delay changes. Mostly there is finite crossings which are finite number of frequency intersections for any given delay threshold [40][41]. For $P_n(\lambda, \tau) = 0$ (characteristic equation) the delay term (τ) not determine the characteristic roots. The roots are simply the zeros of $Q_m(\lambda, \tau)$ and the stability is inspected by Routh-Hurwitz criterion.

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