

Research Article

Spatial Dependence Described by the Solution of the Boltzmann Kinetic Equation

Davy D Tskhakaya

Andronikashvili Institute of Physics (TSU), 0177 Tbilisi, Georgia

Submitted : 27 August, 2026

Accepted : 31 August, 2026

Published : 01 September, 2026

***Corresponding author:** Davy D Tskhakaya, Andronikashvili Institute of Physics (TSU), 0177 Tbilisi, Georgia, E-mail: dtskhak@yahoo.com

Copyright License: © 2026 Tskhakaya DD. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

<https://www.mathematicsgroup.com>



Abstract

For the first time the complete solution of the time-dependent Vlasov-Boltzmann kinetic equation is used to describe waves in a plasma bounded by a cylindrical wall. The dependence of the distribution function (as a general solution of the kinetic equation) on the spatial coordinates is taken into account. A special case of conditions confining plasma in a cylindrical tube is considered analytically.

Introduction

Solving the time-dependent Vlasov-Boltzmann kinetic equation without collisions usually comes down to considering two characteristic equations - the equations for spatial coordinates and for the velocity. Until now, the main attention in published papers has been addressed to the solution of the characteristic equation for velocity - leading to the distribution function for plasma particles. The characteristic equation for spatial coordinates remains unattended. In the manuscript, for the first time, an attempt is made to use the spatial dependence of the particle distribution function for the description of plasma problems.

Apparently, the present paper is the first attempt: 1) to include spatial dependence in the solution of the Vlasov-Boltzmann kinetic equation and 2) to analyze analytically the propagation of longitudinal waves in a cylindrical tube filled with plasma, when the waves propagate along the cylinder axis.

The analysis is carried out within the framework of a characteristic model of conditions for spatial restrictions on the behaviour of plasma. At present, there appears to be no research describing the spatial behaviour of plasma based on directly analytically solving the Vlasov-Boltzmann kinetic equation. The model described in the present manuscript represents and completes an analytic solution of the Vlasov-Boltzmann kinetic equation for electrons and ions. Features of the interaction of plasma particles with the tube wall (reflection, absorption, ...) are so far beyond the scope of this publication.

The paper is organized as follows: in the section 2 the complete solution of the Vlasov-Boltzmann kinetic equation for electrons and ions is analysed; section 3 is devoted to the analysis of the particle distribution function (DF) on velocity and spatial coordinates; plasma waves in a cylindrical tube in section 4 are considered; the propagation of the waves near the axis (section 5) and not far from the wall of a cylindrical pipe (section 6) is described.

Solution of Vlasov-Boltzmann equation

We start from the collisionless kinetic equation [1,2] for electrons and ions,

$$\frac{df_{\alpha}}{dt} = \frac{\partial f_{\alpha}}{\partial t} + \vec{v} \cdot \frac{\partial f_{\alpha}}{\partial \vec{r}} + \frac{z_{\alpha} e}{m_{\alpha}} \left(\vec{E} + [\vec{v} \times \vec{B}] \right) \cdot \frac{\partial f_{\alpha}}{\partial \vec{v}} = 0, \quad \alpha = e, i \quad (1)$$

For ions we assume and for electrons $z_e = -1$. For the solution of Eq. (1) we generalize the method introduced in our previous work [2]. Using well-known relations with vector and scalar potentials $\vec{B} = [\vec{v} \times \vec{A}(\vec{r}, t)]$, and definitions of the following operator, for the expression of the force in the last term of Eq. (1) we obtain:

$$\begin{aligned} \vec{E}(\vec{r}, t) + \frac{1}{c} [\vec{v} \times \vec{B}(\vec{r}, t)] = - \frac{d}{dt} \left\{ \frac{\partial}{\partial \vec{r}} \left[\vec{A}(\vec{r}, t) + \vec{\nabla} \phi(\vec{r}, t) \right] \right\} + \\ + v_i \cdot \vec{\nabla} \cdot \left\{ \int_{t_0}^t dt' \left\{ \frac{\partial}{\partial t'} A_i(\vec{r}, t') + \nabla_i \phi(\vec{r}, t') \right\} \right\}. \end{aligned} \quad (2)$$

Lower limits of integrals will be determined by the assumption that the electric field is switched on at the initial moment t_0 . Introducing the notation,

$$\vec{\varepsilon}(\vec{r}, t) = \int_{t_0}^t dt' \cdot \vec{E}(\vec{r}, t'), \quad (3)$$

the kinetic equation (1) acquires the form

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \frac{\partial f_\alpha}{\partial \vec{r}} + \frac{d}{dt} \cdot \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(\vec{r}, t) \cdot \frac{\partial f_\alpha}{\partial \vec{v}} - \left(v_i \cdot \vec{\nabla} \cdot \frac{z_\alpha e}{m_\alpha} \varepsilon_i(\vec{r}, t) \right) \cdot \frac{\partial f_\alpha}{\partial \vec{v}} = 0. \quad (4)$$

Afterwards, it is convenient to introduce a new velocity variable, $\vec{w} = \vec{v} - \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(\vec{r}, t)$ ($\vec{v} = \vec{w} + \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(\vec{r}, t)$), which implies the change of derivatives in the equation (4):

$$\frac{\partial f_\alpha(t, \vec{r}, \vec{v})}{\partial t} \rightarrow \frac{\partial f_\alpha(t, \vec{r}, \vec{w})}{\partial t} - \frac{\partial}{\partial t} \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(\vec{r}, t) \cdot \frac{\partial f_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{w}}, \quad (5)$$

$$\frac{\partial f_\alpha(t, \vec{r}, \vec{v})}{\partial r_i} \rightarrow \frac{\partial f_\alpha(t, \vec{r}, \vec{w})}{\partial r_i} - \frac{\partial}{\partial r_i} \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(\vec{r}, t) \cdot \frac{\partial f_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{w}}. \quad (6)$$

Using the relation again the equation (4) results in:

$$\frac{\partial f_\alpha}{\partial t} + \left\{ \vec{w} + \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(\vec{r}, t) \right\} \cdot \frac{\partial f_\alpha}{\partial \vec{r}} - \left\{ w_i + \frac{z_\alpha e}{m_\alpha} \varepsilon_i(\vec{r}, t) \right\} \frac{\partial}{\partial \vec{r}} \frac{z_\alpha e}{m_\alpha} \varepsilon_i(\vec{r}, t) \cdot \frac{\partial f_\alpha}{\partial \vec{w}} = 0, \quad (7) \text{ or in compact form } \frac{\partial f_\alpha}{\partial t} + [H_\alpha \cdot f_\alpha] = 0, \quad (8)$$

$$\text{where } [H_\alpha \cdot f_\alpha] = \frac{\partial H_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{w}} \cdot \frac{\partial f_\alpha}{\partial \vec{r}} - \frac{\partial H_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{r}} \cdot \frac{\partial f_\alpha}{\partial \vec{w}} \quad (9)$$

is the Poisson's brackets defined by means of the kinetic energy,

$$H_\alpha(t, \vec{r}, \vec{w}) = \frac{1}{2} \left\{ \vec{w} + \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(t, \vec{r}) \right\}^2 \equiv H_\alpha(t). \quad (10)$$

In the following, for shortening, we sometimes use the notation $H_\alpha(t)$. The characteristic equations for Eq. (8) read

$$\frac{d\vec{w}}{dt} = - \frac{\partial H_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{r}}, \quad (11)$$

$$\frac{d\vec{r}}{dt} = \frac{\partial H_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{w}}. \quad (12)$$

Further calculations are carried out for the characteristic equation only. Results for the characteristic equation for the coordinate $\vec{r}$ (see equation (12)) can be obtained by simple change of notations. From equation (11) we obtain

$$\begin{aligned} \frac{d\vec{w}}{dt} &= - \frac{\partial}{\partial t} \int dt' \frac{\partial H_\alpha(t', \vec{r}, \vec{w})}{\partial \vec{r}} = \\ &= - \frac{d}{dt} \int dt' \frac{\partial H_\alpha(t', \vec{r}, \vec{w})}{\partial \vec{r}} + \left(\frac{d\vec{r}}{dt} \cdot \frac{\partial}{\partial \vec{r}} \right) \int dt' \frac{\partial H_\alpha(t', \vec{r}, \vec{w})}{\partial \vec{r}} + \left(\frac{d\vec{w}}{dt} \cdot \frac{\partial}{\partial \vec{w}} \right) \int dt' \frac{\partial H_\alpha(t', \vec{r}, \vec{w})}{\partial \vec{r}} = \\ &= - \frac{d}{dt} \int dt' \frac{\partial H_\alpha(t')}{\partial \vec{r}} + \left(\frac{dH_\alpha(t)}{d\vec{w}} \cdot \frac{\partial}{\partial \vec{r}} \right) \int dt' \frac{\partial H_\alpha(t')}{\partial \vec{r}} - \left(\frac{dH_\alpha(t)}{d\vec{r}} \cdot \frac{\partial}{\partial \vec{w}} \right) \int dt' \frac{\partial H_\alpha(t')}{\partial \vec{r}}. \end{aligned} \quad (13)$$

Introducing the operator

$$\begin{aligned} \{\hat{t}_\alpha\} &= \frac{d\vec{r}}{dt} \cdot \frac{\partial}{\partial \vec{r}} + \frac{d\vec{w}}{dt} \cdot \frac{\partial}{\partial \vec{w}} = \\ &= \frac{\partial H_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{w}} \cdot \frac{\partial}{\partial \vec{r}} - \frac{\partial H_\alpha(t, \vec{r}, \vec{w})}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{w}}, \end{aligned} \quad (14)$$

and continuing the transformation made for obtaining Eq.(13), we obtain

$$\begin{aligned} \frac{d\vec{w}}{dt} &= - \frac{d}{dt} \int dt' \frac{\partial H_\alpha(t')}{\partial \vec{r}} + \{\hat{t}_\alpha\} \int dt' \frac{\partial H_\alpha(t')}{\partial \vec{r}} = \\ &= - \frac{d}{dt} \int dt' \frac{\partial H_\alpha(t')}{\partial \vec{r}} + \frac{d}{dt} \int dt' \cdot \{\hat{t}_\alpha\} \int dt'' \frac{\partial H_\alpha(t'')}{\partial \vec{r}} - \{\hat{t}_\alpha\} \int dt' \cdot \{\hat{t}_\alpha\} \int dt'' \frac{\partial H_\alpha(t'')}{\partial \vec{r}} \dots \end{aligned} \quad (15)$$

Last term in the right-hand (rh) side of Eq.(15), by analogy to preceding terms, can also be transformed to the form with the full time derivative. The similar procedure can be realized for all subsequent terms. In consequence, we obtain the expression for the first constant of integration for Eq.(1):

$$\vec{U}_\alpha = \vec{w} + \int dt' \frac{\partial H_\alpha(t')}{\partial \vec{r}} - \int dt' \cdot \{\hat{t}_\alpha\} \int dt'' \frac{\partial H_\alpha(t'')}{\partial \vec{r}} + \int dt' \cdot \{\hat{t}_\alpha\} \int dt'' \cdot \{\hat{t}_\alpha\} \int dt''' \frac{\partial H_\alpha(t''')}{\partial \vec{r}} - \quad (16)$$

$$-\int dt' \cdot \{t'_a\} \int dt'' \cdot \{t''_a\} \int dt''' \cdot \{t'''_a\} \int dt'''' \frac{\partial H_a(t'''')}{\partial \vec{r}} + \dots$$

Using the transformation of integrals

$$\int dt' \int dt'' \dots = \int dt'' \int dt' \dots, \quad (17)$$

and after calculations given in Appendix A, the expression (16) acquires the form

$$\bar{U}_a = \bar{w} + \int dt' \left\{ 1 - \int dt'' \cdot \{t''_a\} + \frac{1}{2} \left\{ \int dt'' \cdot \{t''_a\} \right\}^2 - \frac{1}{6} \left\{ \int dt'' \cdot \{t''_a\} \right\}^3 + \dots \right\} \frac{\partial H_a(t', \bar{r}, \bar{w})}{\partial \bar{r}} \quad (18)$$

Expression for the second constant of integration can be obtained by means of a change of variables in equations (12), (13), and (18):

$$\bar{R}_a = \bar{r} - \int dt' \left\{ 1 - \int dt'' \cdot \{t''_a\} + \frac{1}{2} \left\{ \int dt'' \cdot \{t''_a\} \right\}^2 - \frac{1}{6} \left\{ \int dt'' \cdot \{t''_a\} \right\}^3 + \dots \right\} \frac{\partial H_a(t', \bar{r}, \bar{w})}{\partial \bar{w}}. \quad (19)$$

Expressions (18) and (19) represent expansions of the following relations

$$\bar{U}_a = \bar{w} + \int dt' \frac{\partial}{\partial \bar{r}} H_a \left\{ t', \bar{r} - \int dt'' \frac{\partial H_a(t'', \bar{r}, \bar{w})}{\partial \bar{w}}, \bar{w} + \int dt'' \frac{\partial H_a(t'', \bar{r}, \bar{w})}{\partial \bar{r}} \right\}, \quad (20)$$

$$\bar{R}_a = \bar{r} - \int dt' \frac{\partial}{\partial \bar{w}} H_a \left\{ t', \bar{r} - \int dt'' \frac{\partial H_a(t'', \bar{r}, \bar{w})}{\partial \bar{w}}, \bar{w} + \int dt'' \frac{\partial H_a(t'', \bar{r}, \bar{w})}{\partial \bar{r}} \right\}. \quad (21)$$

By direct substitution, one can easily be convinced that the expression,

$$f_a = f_a(\bar{R}_a(t, \bar{r}, \bar{w}), \bar{U}_a(t, \bar{r}, \bar{w})), \quad (22)$$

together with Eqs. (20) and (21) is the solution of the Vlasov-Boltzmann kinetic equation (1). Indeed, from Eqs. (8) and (9) we can construct the identical equation:

$$\frac{\partial f_a}{\partial \bar{R}_a} \left\{ \frac{\partial \bar{R}_a}{\partial t} + \frac{\partial H_a}{\partial \bar{w}} \frac{\partial \bar{R}_a}{\partial \bar{r}} - \frac{\partial H_a}{\partial \bar{r}} \frac{\partial \bar{R}_a}{\partial \bar{w}} \right\} + \frac{\partial f_a}{\partial \bar{U}_a} \left\{ \frac{\partial \bar{U}_a}{\partial t} + \frac{\partial H_a}{\partial \bar{w}} \frac{\partial \bar{U}_a}{\partial \bar{r}} - \frac{\partial H_a}{\partial \bar{r}} \frac{\partial \bar{U}_a}{\partial \bar{w}} \right\} = 0. \quad (23)$$

Taking derivatives with time from equations (20) and (21), we find, $\frac{\partial \bar{R}_a}{\partial t} + \{t'_a\} \cdot \bar{R}_a = 0$, and $\frac{\partial \bar{U}_a}{\partial t} + \{t'_a\} \cdot \bar{U}_a = 0$. (24) Hence, the distribution function then should be presented in the form (22). For further analysis, we need to determine inverse solutions for $\bar{r}$ and $\bar{w}$ of Eqs. (20) and (21). From these equations it follows that

$$\bar{r} - \int dt' \frac{\partial H_a(t', \bar{r}, \bar{w})}{\partial \bar{w}} = \bar{\rho}_a(t) = \bar{R}_a + \int dt' \frac{\partial}{\partial \bar{w}} H_a \{ \bar{\rho}_a(t'), \bar{\omega}_a(t') \}, \quad (25)$$

$$\bar{r} - \int dt'' \frac{\partial H_a(t'', \bar{r}, \bar{w})}{\partial \bar{w}} = \bar{\rho}_a(t') = \bar{R}_a + \int dt'' \frac{\partial}{\partial \bar{w}} H_a \{ \bar{\rho}_a(t''), \bar{\omega}_a(t'') \}, \quad (25')$$

$$\bar{r} - \int dt''' \frac{\partial H_a(t''', \bar{r}, \bar{w})}{\partial \bar{w}} = \bar{\rho}_a(t'') = \bar{R}_a + \int dt''' \frac{\partial}{\partial \bar{w}} H_a \{ \bar{\rho}_a(t'''), \bar{\omega}_a(t''') \}, \quad (25'')$$

$$\bar{w} + \int dt' \frac{\partial H_a(t', \bar{r}, \bar{w})}{\partial \bar{r}} = \bar{\omega}_a(t) = \bar{U}_a - \int dt' \frac{\partial}{\partial \bar{r}} H_a \{ \bar{\rho}_a(t'), \bar{\omega}_a(t') \}, \quad (26)$$

$$\bar{w} + \int dt'' \frac{\partial H_a(t'', \bar{r}, \bar{w})}{\partial \bar{r}} = \bar{\omega}_a(t') = \bar{U}_a - \int dt'' \frac{\partial}{\partial \bar{r}} H_a \{ \bar{\rho}_a(t''), \bar{\omega}_a(t'') \}, \quad (26')$$

$$\bar{w} + \int dt''' \frac{\partial H_a(t''', \bar{r}, \bar{w})}{\partial \bar{r}} = \bar{\omega}_a(t'') = \bar{U}_a - \int dt''' \frac{\partial}{\partial \bar{r}} H_a \{ \bar{\rho}_a(t'''), \bar{\omega}_a(t''') \}, \quad (26'')$$

The chain of expressions (25, 25', ...) and (26, 26', ...) can be continued, which allows for presenting inverse solutions in the following form:

$$\bar{r} = \bar{R} + \int dt' \frac{\partial}{\partial \bar{U}} H \left\{ t', \bar{R} + \int dt'' \frac{\partial}{\partial \bar{U}} H \{ t'', \bar{\rho}(t''), \bar{\omega}(t'') \}, \bar{U} - \int dt'' \frac{\partial}{\partial \bar{R}} H \{ t'', \bar{\rho}(t''), \bar{\omega}(t'') \} \right\}, \quad (27)$$

$$\bar{w} = \bar{U} - \int dt' \frac{\partial}{\partial \bar{R}} H \left\{ t', \bar{R} + \int dt'' \frac{\partial}{\partial \bar{U}} H \{ t'', \bar{\rho}(t''), \bar{\omega}(t'') \}, \bar{U} - \int dt'' \frac{\partial}{\partial \bar{R}} H \{ t'', \bar{\rho}(t''), \bar{\omega}(t'') \} \right\}. \quad (28)$$

By successively substituting and $\bar{\omega}(t''), \bar{\omega}(t'''), \dots$ into solutions (27) and (28), these solutions take the form of an infinite series of the constants of integration, $\bar{R}$ and $\bar{U}$, in the arguments of functions $H \{ t'', \bar{\rho}(t''), \bar{\omega}(t'') \}, H \{ t''', \bar{\rho}(t'''), \bar{\omega}(t''') \}, \dots$. Solutions (27) and (28) we'll use below.

Distribution functions for particles

Distribution functions we define assuming that at the initial moment, the electron and ion distribution functions (DFs) depend only on the spatial coordinate and the velocity expressed through integration constants $\bar{R}_\alpha$ and $\bar{U}_\alpha$,

$$f_\alpha \left\{ \bar{R}_\alpha(t_0, \vec{r}, \vec{v}), \bar{U}_\alpha(t_0, \vec{r}, \vec{v}) \right\} = f_\alpha(\vec{r}, \vec{v}) \Big|_{t=t_0}. \quad (29)$$

From the relations at the initial time t_0 it follows:

$$\bar{R}_\alpha(t_0, \vec{r}, \vec{w}) = \bar{R}_\alpha \text{ and } \bar{U}_\alpha(t_0, \vec{r}, \vec{w}) = \bar{U}_\alpha, \quad (30)$$

where, and is determined by Eq. (3). Then, by means of Eqs. (27) and (28) we can find an expression for the coordinate $\vec{r}$ and the velocity $\vec{v}$:

$$\vec{r} = \quad (31)$$

$$= \bar{R}_\alpha + \int_{t_0}^t dt' \frac{\partial}{\partial \bar{U}_\alpha} H_\alpha \left\{ \bar{R}_\alpha + \int_{t_0}^{t'} dt'' \frac{\partial}{\partial \bar{U}_\alpha} H_\alpha \left\{ \bar{R}_\alpha + \dots, \bar{U}_\alpha - \dots \right\}, \bar{U}_\alpha - \int_{t_0}^{t'} dt'' \frac{\partial}{\partial \bar{R}_\alpha} H_\alpha \left\{ \bar{R}_\alpha + \dots, \bar{U}_\alpha - \dots \right\} \right\}$$

$$\vec{v} = \quad (32)$$

$$= \bar{U}_\alpha - \int_{t_0}^t dt' \frac{\partial}{\partial \bar{R}_\alpha} H_\alpha \left\{ \bar{R}_\alpha + \int_{t_0}^{t'} dt'' \frac{\partial}{\partial \bar{U}_\alpha} H_\alpha \left\{ \bar{R}_\alpha + \dots, \bar{U}_\alpha - \dots \right\}, \bar{U}_\alpha - \int_{t_0}^{t'} dt'' \frac{\partial}{\partial \bar{R}_\alpha} H_\alpha \left\{ \bar{R}_\alpha + \dots, \bar{U}_\alpha - \dots \right\} \right\}.$$

Substituting the relations (20) and (21) into (31) and (32), and then the whole right sides of expressions (31) and (32) (while recalling the definition of $\vec{\varepsilon}(\vec{r}, t)$ (see Eq. (3) and expression for $\vec{w}$)),

$$\vec{r} - \int_{t_0}^t dt' \frac{\partial}{\partial \vec{w}} H_\alpha \left\{ t', \vec{r} - \int_{t_0}^{t'} dt'' \frac{\partial H_\alpha(t'', \vec{r}, \vec{w})}{\partial \vec{w}}, \vec{w} + \int_{t_0}^{t'} dt'' \frac{\partial H_\alpha(t'', \vec{r}, \vec{w})}{\partial \vec{r}} \right\} = \bar{\Gamma}_\alpha, \quad (33)$$

$$\vec{w} + \int_{t_0}^t dt' \frac{\partial}{\partial \vec{r}} H_\alpha \left\{ t', \vec{r} - \int_{t_0}^{t'} dt'' \frac{\partial H_\alpha(t'', \vec{r}, \vec{w})}{\partial \vec{w}}, \vec{w} + \int_{t_0}^{t'} dt'' \frac{\partial H_\alpha(t'', \vec{r}, \vec{w})}{\partial \vec{r}} \right\} = \bar{W}_\alpha, \quad (34)$$

into Eq. (29), for the particle distribution functions (DFs) we obtain

$$f_\alpha = f_\alpha \left\{ \bar{\Gamma}_\alpha(t, \vec{r}, \vec{w}), \bar{W}_\alpha(t, \vec{r}, \vec{w}) \right\} = f_\alpha \left\{ \bar{\Gamma}_\alpha, \bar{W}_\alpha \right\}. \quad (35)$$

For further analysis, we represent this distribution function in the form,

$$f_\alpha \left\{ \bar{\Gamma}_\alpha, \bar{W}_\alpha \right\} = P_\alpha \left\{ \bar{\Gamma}_\alpha(\vec{r}) \right\} \cdot V_\alpha \left\{ \vec{w} \right\}, \quad (36)$$

and discuss each of the multipliers separately.

i) Dependence on the velocity.

At the initial moment, we will assume that the velocity distribution function obeys the Maxwell distribution:

$$V_\alpha(v^2) = n_{\alpha 0} \left(m_\alpha / (2\pi T_\alpha) \right)^{3/2} \cdot \exp \left(- m_\alpha v^2 / 2T_\alpha \right). \quad (37)$$

According to Eq. (34), it can apparently be assumed that for later times the distribution functions are expressed by the equation:

$$V_\alpha(w^2) = n_{\alpha 0} \left(m_\alpha / (2\pi T_\alpha) \right)^{3/2} \cdot \exp \left\{ - \frac{m_\alpha}{2T_\alpha} W_\alpha^2 \right\} = n_{\alpha 0} \left(m_\alpha / (2\pi T_\alpha) \right)^{3/2} \cdot \quad (38)$$

$$\cdot \exp \left\{ - \frac{m_\alpha}{2T_\alpha} \left[\vec{w} + \int_{t_0}^t dt' \frac{\partial}{\partial \vec{r}} H_\alpha \left\{ t', \vec{r} - \int_{t_0}^{t'} dt'' \frac{\partial H_\alpha(t'', \vec{r}, \vec{w})}{\partial \vec{w}}, \vec{w} + \int_{t_0}^{t'} dt'' \frac{\partial H_\alpha(t'', \vec{r}, \vec{w})}{\partial \vec{r}} \right\} \right]^2 \right\}.$$

In what follows, we will limit ourselves to preserving only the first powers of the weak electric field (that is, $\vec{\varepsilon}(\vec{r}, t)$). Assuming the electric field to be weak, we can expand in Eq. (38) the exponential function by degrees of the field. Thus, taking into account Eqs. (18) and (20) lead to following results (see Appendix B) :

$$V_\alpha(w^2) = n_{\alpha 0} \left(m_\alpha / (2\pi T_\alpha) \right)^{3/2} \cdot \exp \left\{ - \frac{m_\alpha}{2T_\alpha} w^2 \right\} \left\{ 1 - \frac{m_\alpha}{T_\alpha} \Phi_\alpha(\vec{w}) \right\}, \quad (39)$$

$$\Phi_\alpha(\vec{w}) = \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \int_{t_0}^t dt' \left(\vec{w} \cdot \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(t', \vec{r}) \right) - \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \left(\vec{w} \cdot \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(t', \vec{r}) \right) + \quad (40)$$

$$+ \frac{1}{2} \cdot \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \left(\vec{w} \cdot \frac{z_\alpha e}{m_\alpha} \vec{\varepsilon}(t', \vec{r}) \right).$$

Obviously, in subsequent calculations, as a result of integration over velocity, the second term on the right-hand side of Eqs. (B.3) and (40), containing "w" to an odd degree, disappears.

ii) Multiplier describing the spatial dependence of the DF

Let us consider the case when the plasma is placed in a cylindrical tube with a radius, and the waves propagate along the z - axis of the cylinder (see Figure 1).

According to Eq. (33), the cylindrical dependence of the spatial multiplier $P_a\{\bar{\Gamma}_a(\bar{r})\}$ on the coordinate $\mathbf{r}$, at the initial moment, can be represented in the form

$$P_a\{\Gamma_a\}_{t_0} = P_a\{r\} = P_a\left\{\sqrt{x^2 + y^2}\right\}. \text{ For later time, according to (C.1) (see Appendix C), we have } P_a\{\Gamma_a\} = P_a\{\bar{r} - \bar{\mathfrak{S}}_a\} = P_a\left\{\sqrt{(x - \mathfrak{S}_{ax})^2 + (y - \mathfrak{S}_{ay})^2}\right\}, \quad (41)$$

where $\mathfrak{S}_x$ and $\mathfrak{S}_y$ are defined according to (C.2). Remembering the presence of a weak electric field, the expression (41) can be simplified (keeping the expression for the electric field only to the first power),

$$P_a\{\Gamma_a\} = P_a\{\bar{r} - \bar{\mathfrak{S}}_a\} = P_a\left\{r\sqrt{1 - 2\frac{\bar{r} \cdot \bar{\mathfrak{S}}_a}{r^2} + \frac{\bar{\mathfrak{S}}_a^2}{r^2}}\right\} \cong \quad (42)$$

$$\cong P(r) - \left\{ \frac{\bar{r} \cdot \bar{\mathfrak{S}}_a}{r} - \frac{1}{2 \cdot r} \left(\bar{\mathfrak{S}}_a^2 - \frac{(\bar{r} \cdot \bar{\mathfrak{S}}_a)^2}{r^2} \right) \right\} \cdot \frac{\partial}{\partial r} P(r),$$

where the vector has only two radial components, and we have neglected terms of the order $(\mathfrak{S}/r)^2$. In the presence of a multiplier $\frac{\partial}{\partial r} P(r)$ (see Eq. (43)) and with a considerable value of the cylinder radius, the value $(\mathfrak{S}/r)$ can be considered small. The cylinder, confining the plasma, can be described by the relation

$$P(r) = \Theta(r_0 - r) + \frac{r_s - r}{r_s - r_0} \Theta(r - r_0) \cdot \Theta(r_s - r), \quad (43)$$

where the general definition of the *Heaviside* step function is given by the relation

$$\Theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}. \quad (44)$$

In fact, the multiplier (43) defines the radial shape of plasma characteristics. The wall of the cylinder is located at a distance from the axis of the cylinder. Here we will not discuss the structure of the layer adjoining the wall. Considering the thickness of this layer to be small, the profile of the special factor for the wall layer we have chosen in the form,

$$q_s = \frac{r_s - r}{r_s - r_0}, \quad (45)$$

describing the linear reduction of the area under consideration to the bounded wall; r_s is an auxiliary point for describing this descending area (r_s is outside of the pipe).

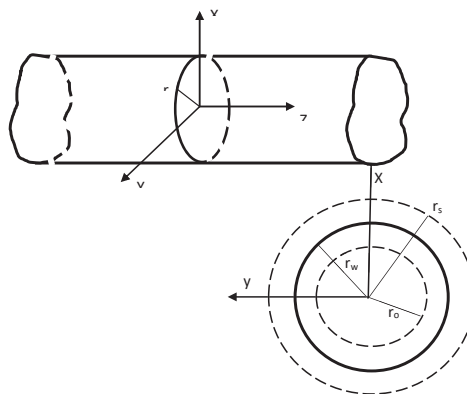


Figure 1: Schematic view of a cylindrical tube and the main characteristic distances (and radii) in the cross section of a cylindrical pipe: r_w is the radius of the pipe, point r_0 is the limit of the homogeneous region of the spatial multiplier, point r_s is located outside of the pipe and at this point the profile of the spatial multiplier becomes zero (see also Figure 2).

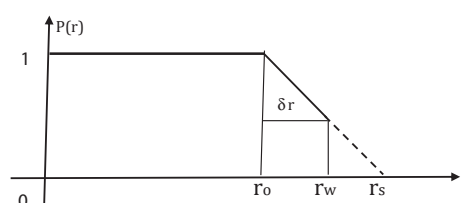


Figure 2: Dependence $P(r)$ on the radial coordinate (r).

Plasma waves inside a cylindrical tube

We will start with the equation commonly used to treat longitudinal waves,

$$\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) = \sum_{\alpha=e,i} 4\pi \cdot z_{\alpha} e \cdot \int d\vec{v} \cdot f_{\alpha}(t, \vec{r}, \vec{v}) = \quad (46)$$

$= \sum_{\alpha=e,i} 4\pi \cdot z_{\alpha} e \cdot \int d\vec{v} \cdot P_{\alpha} \{ \Gamma_{\alpha} \} \cdot V_{\alpha} \{ \vec{w} \}$. Substituting expressions (39) and (42) into Eq. (46), we obtain

$$\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) = \sum_{\alpha=e,i} 4\pi \cdot z_{\alpha} e \cdot n_{0\alpha} \left(\frac{m_{\alpha}}{2\pi T_{\alpha}} \right)^{\frac{3}{2}} \cdot \int d\vec{w} \cdot \exp \left\{ -\frac{m_{\alpha}}{2T_{\alpha}} w^2 \right\} \cdot \left\{ 1 - \frac{m_{\alpha}}{T_{\alpha}} \Phi_{\alpha}(\vec{w}) \right\} \cdot \left\{ P(r) - \left[\frac{\vec{r} \cdot \vec{\Sigma}_{\alpha}}{r} - \frac{1}{2 \cdot r} \left(\Sigma_{\alpha}^2 - \frac{(\vec{r} \cdot \vec{\Sigma}_{\alpha})^2}{r^2} \right) \right] \cdot \frac{\partial}{\partial r} P(r) \right\} \quad (47)$$

Here we replaced integration over $\vec{v}$ with the integration over $\vec{w}$. Considering that in unperturbed plasma (in a plasma without an electric field) quasi-neutrality is preserved,

$$\sum_{\alpha=e,i} z_{\alpha} n_{0\alpha} = 0. \quad (48)$$

after multiplying terms in curly brackets of (47), the first term drops out according to (48). Taking into account the electric field only to the first power, from (47) we obtain

$$\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) = - \sum_{\alpha=e,i} 4\pi \cdot z_{\alpha} e \cdot n_{0\alpha} \left(\frac{m_{\alpha}}{2\pi T_{\alpha}} \right)^{\frac{3}{2}} \cdot \int d\vec{w} \cdot \exp \left\{ -\frac{m_{\alpha}}{2T_{\alpha}} w^2 \right\} \cdot \left\{ P(r) \frac{m_{\alpha}}{T_{\alpha}} \Phi_{\alpha}(\vec{w}) + \left(\frac{\vec{r} \cdot \vec{\Sigma}_{\alpha}}{r} - \frac{1}{2 \cdot r} \left(\Sigma_{\alpha}^2 - \frac{(\vec{r} \cdot \vec{\Sigma}_{\alpha})^2}{r^2} \right) \right) \frac{\partial}{\partial r} P(r) \right\}. \quad (49)$$

$$\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) = - \sum_{\alpha} \omega_{p\alpha}^2 \left\{ P(r) \left[\int_{t_0}^t dt' \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t') + \frac{3}{2} V_{T\alpha}^2 \Delta \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t''') \right] + \right. \quad (50)$$

$$\left. + \left[\int_{t_0}^t dt' \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t') + \frac{1}{2} V_{T\alpha}^2 \Delta \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t''') \right] \cdot \frac{\partial}{\partial r} P(r) - \frac{m_{\alpha}}{z_{\alpha} \cdot e} \frac{T_{\alpha}}{m_{\alpha}} \frac{\partial}{\partial r} P(r) \right\},$$

where and $V_{T\alpha} = \sqrt{T_{\alpha}/m_{\alpha}}$. In Eq. (50), we have used the relation

$$\frac{\partial}{\partial \vec{r}} P(r) = \frac{\vec{r}}{r} \frac{\partial}{\partial r} P(r), \quad (51)$$

and the following integrations are performed:

$$\int d\vec{u} \cdot u_k u_k \cdot \exp(-u^2/2) = (2\pi)^{\frac{3}{2}} \cdot \delta_k, \quad (52)$$

$$\int d\vec{u} \cdot u_k u_k u_n \frac{\partial}{\partial r_l} \frac{\partial}{\partial r_k} \frac{\partial}{\partial r_n} (\vec{u} \vec{\sigma}) \cdot \exp(-u^2/2) = (2\pi)^{\frac{3}{2}} \cdot \frac{3}{2} \cdot \frac{\partial}{\partial r_l} \frac{\partial}{\partial r_k} \left(\frac{\partial \sigma_l}{\partial r_k} + \frac{\partial \sigma_k}{\partial r_l} \right). \quad (53)$$

The right-hand side of Eq. (53) for longitudinal waves gives $(2\pi)^{\frac{3}{2}} \cdot 3 \cdot \Delta \cdot \text{div} \vec{\sigma}$. The second derivative of expression (50) with time is equal to

$$\frac{\partial^2}{\partial t^2} \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) = - \sum_{\alpha} \omega_{p\alpha}^2 \left\{ P(r) \left[\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) + 3 \cdot V_{T\alpha}^2 \Delta \int_{t_0}^t dt' \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t') \right] + \right. \quad (54)$$

$$\left. + \left[\vec{E}(\vec{r}, t) + V_{T\alpha}^2 \Delta \int_{t_0}^t dt' \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t') \right] \cdot \frac{\partial}{\partial r} P(r) - \frac{2T_{\alpha}}{z_{\alpha} \cdot e} \frac{1}{r} \frac{\partial}{\partial r} P(r) \right\}.$$

Recalling the definition (3) of the value $\vec{E}(\vec{r}, t)$ and using (54), for the electric field we obtain:

$$\frac{\partial^4}{\partial t^4} \frac{\partial \vec{E}}{\partial \vec{r}} + \sum_{\alpha} \omega_{p\alpha}^2 P(r) \left[\frac{\partial^2}{\partial t^2} \frac{\partial \vec{E}}{\partial \vec{r}} + 3 \cdot V_{T\alpha}^2 \Delta \frac{\partial \vec{E}}{\partial \vec{r}} \right] = - \sum_{\alpha} \omega_{p\alpha}^2 \left[\frac{\partial^2 \vec{E}}{\partial t^2} + V_{T\alpha}^2 \Delta \cdot \vec{E} \right] \frac{\partial}{\partial \vec{r}} P(r). \quad (55)$$

or, after a simple transformation of the right-hand side:

$$\frac{\partial}{\partial \vec{r}} \cdot \left[\frac{\partial^4}{\partial t^4} + \sum_{\alpha} \omega_{p\alpha}^2 P(r) \left[\frac{\partial^2}{\partial t^2} + 3 \cdot V_{T\alpha}^2 \Delta \right] \right] \vec{E} = \sum_{\alpha} \omega_{p\alpha}^2 \frac{\partial}{\partial \vec{r}} P(r) \cdot 2V_{T\alpha}^2 \Delta \cdot \vec{E}. \quad (56)$$

Below we will look at plasma waves in a cylindrical pipe.

Waves inside the pipe

In this case, we assume that the waves propagate in the vicinity of the cylinder axis in the direction of the z-axis. According to the expressions (43), the spatial value $P(r) = \Theta(r_0 - r)$ at $r < r_0$ is equal to 1, and $(\partial/\partial r)P(r)$ at $r < r_0$ is equal to zero, as $(\partial/\partial r)P(r) = -\delta(r_0 - r)$ (see Eq. (43)). Neglecting terms with ion temperature, Eq. (56) becomes simplified and we obtain:

$$\left\{ \frac{\partial^4}{\partial t^4} + (\omega_p^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + 3 \cdot \omega_p^2 \cdot V_{Te}^2 \Delta \right\} \bar{E} = 0. \quad (57)$$

For waves propagating along the z-axis, are described by the equation:

$$E_z = a \cdot \sin(\omega \cdot t - k_z \cdot z). \quad (58)$$

For the frequency from Eq. (57) we have [3,4]:

$$\omega = \omega_p \left(1 + \omega_p^2 / \omega_{pi}^2 + 3 \cdot k_z^2 \cdot r_D^2 \right)^{\frac{1}{2}} \approx \omega_p \left(1 + \frac{1}{2} \frac{\omega_p^2}{\omega_{pi}^2} \cdot 3 \cdot k_z^2 \cdot r_D^2 \right). \quad (59)$$

Waves on the boundary wall

We can represent Eq. (56) in the form:

$$\frac{\partial}{\partial x} \cdot \left\{ \frac{\partial^4}{\partial t^4} + P(r) \left[(\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + \omega_{pe}^2 \cdot V_{Te}^2 \Delta \right] \right\} E_x + \quad (60)$$

$$+ \frac{\partial}{\partial y} \cdot \left\{ \frac{\partial^4}{\partial t^4} + P(r) \left[(\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + \omega_{pe}^2 \cdot V_{Te}^2 \Delta \right] \right\} E_y +$$

$$+ \frac{\partial}{\partial z} \cdot \left\{ \frac{\partial^4}{\partial t^4} + P(r) \left[(\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + 3 \cdot \omega_{pe}^2 \cdot V_{Te}^2 \Delta \right] \right\} E_z =$$

$$= -\omega_{pe}^2 \cdot P(r) \cdot 2V_{Te}^2 \cdot \Delta \left\{ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right\}. \quad (61)$$

After simply transforming the right-hand side of Eq. (60) and using the relation for cylindrical symmetry,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{1}{r} \frac{\partial}{\partial r} E_r + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi}, \quad (61)$$

(of course, the second term on the right-hand side of Eq. (61) drops out due to cylindrical symmetry), we find

$$\frac{1}{r} \frac{\partial}{\partial r} r \cdot \left\{ \frac{\partial^4}{\partial t^4} + P(r) \left[(\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + \omega_{pe}^2 \cdot V_{Te}^2 \Delta \right] \right\} E_r + \quad (62)$$

$$+ \frac{\partial}{\partial z} \cdot \left\{ \frac{\partial^4}{\partial t^4} + P(r) \left[(\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + 3 \cdot \omega_{pe}^2 \cdot V_{Te}^2 \Delta \right] \right\} E_z = -\omega_{pe}^2 \cdot P(r) \cdot 2V_{Te}^2 \cdot \Delta \cdot \frac{1}{r} \frac{\partial}{\partial r} r E_r.$$

In the cylindrical model under consideration, we can assume that the radial component of the field is equal to zero inside the cylinder, as the radial gradient is absent. Due to the presence of broken places, the selected profile (43), shown in Fig.2, rather roughly describes the properties of plasma confinement. To bypass broken places in the boundaries of integration, we will replace with $r_0 + \varepsilon$, where $\varepsilon \rightarrow +0$ (and this replacement will be implied further). We integrate the near-wall region over a thin layer (r_0, r_w) , with a thickness δr ($\delta r < r_0$), assuming $r_w = r_0 + \delta r$. In the cylindrical variables, this integration takes the form:

$$\begin{aligned} & \int_0^{2\pi} d\phi \int_{r_0}^{r_0+\delta r} r dr \cdot \frac{1}{r} \frac{\partial}{\partial r} r \cdot \left\{ \frac{\partial^4}{\partial t^4} + \omega_{pe}^2 P(r) \left[(\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + \omega_{pe}^2 \cdot V_{Te}^2 \Delta \right] \right\} E_r(t, r, z) + \\ & + \int_0^{2\pi} d\phi \int_{r_0}^{r_0+\delta r} r dr \cdot \frac{\partial}{\partial z} \left\{ \frac{\partial^4}{\partial t^4} + P(r) \cdot \left[(\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + 3 \cdot V_{Te}^2 \Delta \right] \right\} E_z(t, r, z) = \\ & = -\omega_{pe}^2 \cdot \int_0^{2\pi} d\phi \int_{r_0}^{r_0+\delta r} r dr \cdot P(r) \cdot 2V_{Te}^2 \Delta \cdot \frac{1}{r} \frac{\partial}{\partial r} r E_r(t, r, z). \end{aligned} \quad (63)$$

Expanding the integrals (63) in a series in terms of degree δr and taking into account the relation,

$$\begin{aligned} P(r_0) &= \Theta(r_0 - r_0 - \varepsilon) + \frac{r_s - r_0 - \varepsilon}{r_s - r_0} \cdot \Theta(r_0 + \varepsilon - r_0) \cdot \Theta(r_s - r_0) = \\ &= \Theta(-\varepsilon) + 1 \cdot \Theta(+\varepsilon) \cdot 1 = 1, \end{aligned} \quad (64)$$

following from the definition (44), in the first approximation we obtain,

$$\begin{aligned} & 2\pi \cdot \delta r \cdot \frac{\partial}{\partial r} r \cdot \left\{ \frac{\partial^4}{\partial t^4} + \left[(\omega_p^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + \omega_p^2 \cdot V_{Te}^2 \Delta \right] \right\} E_r(t, r, z) \Big|_{r=r_0} + \\ & + 2\pi \cdot \delta r \cdot r \cdot \frac{\partial}{\partial z} \left\{ \frac{\partial^4}{\partial t^4} + \left[(\omega_p^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + 3 \cdot V_{Te}^2 \Delta \right] \right\} E_z(t, r, z) \Big|_{r=r_0} = \\ & = -2\pi \cdot \delta r \cdot r \cdot \omega_p^2 \cdot 2V_{Te}^2 \Delta \cdot \frac{1}{r} \frac{\partial}{\partial r} r E_r(t, r, z) \Big|_{r=r_0}. \end{aligned} \quad (65)$$

Collecting all the terms on the left-hand side of the equation and introducing the electron density as above,

$$\frac{1}{r} \frac{\partial}{\partial r} r E_r + \frac{\partial}{\partial z} E_z = \frac{\partial \bar{E}}{\partial r} = -4\pi \cdot e \delta n_e, \quad (66)$$

we obtain the equation,

$$\left\{ \frac{\partial^4}{\partial t^4} + (\omega_{pe}^2 + \omega_{pi}^2) \frac{\partial^2}{\partial t^2} + 3 \cdot V_{Te}^2 \frac{\partial^2}{\partial z^2} \right\} \cdot \delta n_e = 0, \quad (67)$$

which describes the longitudinal wave, $\delta n_e = A \cdot \sin(\omega \cdot t - k_z \cdot z)$, propagating along the boundary of the cylinder with a frequency:

$$\omega = \omega_p \left(1 + \omega_p^2 / \omega_p^2 + 3 \cdot k_z^2 \cdot r_D^2 \right)^{\frac{1}{2}} \approx \omega_p \left(1 + (1/2) \cdot \omega_p^2 / \omega_p^2 + (3/2) \cdot k_z^2 \cdot r_D^2 \right). \quad (68)$$

Thus the character (dependences on time and coordinates) and frequency of waves propagating near the wall along the axis of a cylindrical pipe do not change (see also Eqs. (57)-(59)).

Summary

The analytic solution of the time-dependent Vlasov-Boltzmann kinetic equation, found in the present paper, in addition to time, also depends on spatial coordinates and velocity, which are solutions of characteristic equations for coordinates and velocity. In publications, in overwhelming cases, the dependence of the distribution function (DF) on velocity is discussed and used. In this paper, probably for the first time, an attempt is made to describe the coordinate dependence of the DF, obtained directly from the kinetic equation.

We consider plasma placed in a cylindrical confining tube. To simplify the problem, we consider the case when the plasma characteristics drop sharply in the radial direction at the boundary. Wave propagation near the cylinder axis and close to the surface of the tube is described.

Consideration of the case when plasma characteristics smoothly diminish to boundary values and analysis of the interaction of plasma particles with the wall require more careful and detailed consideration.

Appendix A

We start from the expression (16);

$$\begin{aligned} \bar{U} &= \bar{w} + \int dt' \frac{\partial H(t')}{\partial \bar{r}} - \int dt'' \int dt' \{t'\} \frac{\partial H(t'')}{\partial \bar{r}} + \int dt' \{t'\} \int dt'' \{t''\} \frac{\partial H(t''')}{\partial \bar{r}} - \\ &- \int dt' \{t'\} \int dt'' \{t''\} \int dt''' \{t'''\} \frac{\partial H(t''')}{\partial \bar{r}} + \dots \\ \bar{w} &= \int dt' \frac{\partial H(t')}{\partial \bar{r}} - \int dt'' \int dt' \{t'\} \frac{\partial H(t'')}{\partial \bar{r}} + \int dt''' \int dt' \{t'\} \int dt'' \{t''\} \frac{\partial H(t''')}{\partial \bar{r}} - \\ &= - \int dt' \{t'\} \int dt'' \{t''\} \int dt''' \{t'''\} \frac{\partial H(t''')}{\partial \bar{r}} + \dots \\ &= \bar{w} + \int dt' \frac{\partial H(t')}{\partial \bar{r}} - \int dt'' \int dt' \{t'\} \frac{\partial H(t'')}{\partial \bar{r}} + \int dt''' \int dt' \{t'\} \int dt'' \{t''\} \frac{\partial H(t''')}{\partial \bar{r}} - \\ &- \int dt''' \int dt' \{t'\} \int dt'' \{t''\} \int dt'''' \{t''''\} \frac{\partial H(t''')}{\partial \bar{r}} + \dots \\ &= w + \int dt' \frac{\partial H(t')}{\partial \bar{r}} - \int dt'' \int dt' \{t'\} \frac{\partial H(t'')}{\partial \bar{r}} + \int dt''' \int dt' \frac{\partial}{\partial t'} \int dt'' \{t''\} \cdot \int dt'' \{t''\} \frac{\partial H(t''')}{\partial \bar{r}} - \\ &- \int dt''' \int dt' \{t'\} \int dt'' \frac{\partial}{\partial t''} \int dt'''' \{t''''\} \int dt'''' \{t''''\} \frac{\partial H(t''')}{\partial \bar{r}} + \dots \\ &= w + \int dt' \frac{\partial H(t')}{\partial \bar{r}} - \int dt'' \int dt' \{t'\} \frac{\partial H(t'')}{\partial \bar{r}} + \int dt''' \int dt' \frac{1}{2} \frac{\partial}{\partial t'} \left\{ \int dt'' \{t''\} \right\}^2 \frac{\partial H(t''')}{\partial \bar{r}} - \\ &- \int dt''' \int dt' \{t'\} \int dt'' \frac{1}{2} \frac{\partial}{\partial t''} \left\{ \int dt'''' \{t''''\} \right\}^2 \frac{\partial H(t''')}{\partial \bar{r}} + \dots \\ &= w + \int dt' \frac{\partial H(t')}{\partial \bar{r}} - \int dt'' \int dt' \{t'\} \frac{\partial H(t'')}{\partial \bar{r}} + \int dt''' \frac{1}{2} \left\{ \int dt'' \{t''\} \right\}^2 \frac{\partial H(t''')}{\partial \bar{r}} - \\ &- \int dt''' \int dt' \frac{\partial}{\partial t'} \int dt'' \{t''\} \cdot \frac{1}{2} \left\{ \int dt'''' \{t''''\} \right\}^2 \frac{\partial H(t''')}{\partial \bar{r}} + \dots \end{aligned}$$

By renaming the variables $t''' \rightarrow t'$, $t'' \rightarrow t'$, ... and so on, we obtain expression (18),

$$\bar{U} = \bar{w} + \int dt' \left\{ 1 - \int dt'' \cdot \{t''\} + \frac{1}{2} \left(\int dt'' \cdot \{t''\} \right)^2 - \frac{1}{6} \left(\int dt'' \cdot \{t''\} \right)^3 + \dots \right\} \frac{\partial H(t', z, w)}{\partial \bar{r}}. \quad (18)$$

Appendix B

Let's start with Eq. (34) written in a form more convenient for analysis

$$\bar{W}_a = \bar{w} + \int_{t_0}^t dt' \left\{ 1 - \int dt'' \cdot \{t''_a\} + \frac{1}{2} \left(\int dt'' \cdot \{t''_a\} \right)^2 - \frac{1}{6} \left(\int dt'' \cdot \{t''_a\} \right)^3 + \dots \right\} \frac{\partial H_a(t', \bar{r}, \bar{w})}{\partial \bar{r}}. \quad (B.1)$$

Using expression (14) for the square of the value, $\bar{w}_a$ we find

$$W_a^2 = w^2 + \left(\int_{t_0}^t dt' \frac{\partial H(t')}{\partial \vec{r}} \right)^2 + \left(\int_{t_0}^t dt' \int_{t'}^t dt'' \left\{ \frac{\partial H(t'')}{\partial \vec{w}} \frac{\partial}{\partial \vec{r}} - \frac{\partial H(t'')}{\partial \vec{r}} \frac{\partial}{\partial \vec{w}} \right\} \cdot \frac{\partial H(t')}{\partial \vec{r}} \right)^2 + \quad (B.2)$$

$$+ 2\vec{w} \cdot \int_{t_0}^t dt' \frac{\partial H(t')}{\partial \vec{r}} - 2\vec{w} \cdot \int_{t_0}^t dt' \int_{t'}^t dt'' \left\{ \frac{\partial H(t'')}{\partial \vec{w}} \frac{\partial}{\partial \vec{r}} - \frac{\partial H(t'')}{\partial \vec{r}} \frac{\partial}{\partial \vec{w}} \right\} \cdot \frac{\partial H(t')}{\partial \vec{r}} +$$

$$+ \vec{w} \cdot \int_{t_0}^t dt' \left(\int_{t'}^t dt'' \left\{ \frac{\partial H(t'')}{\partial \vec{w}} \frac{\partial}{\partial \vec{r}} - \frac{\partial H(t'')}{\partial \vec{r}} \frac{\partial}{\partial \vec{w}} \right\} \right)^2 \cdot \frac{\partial H(t')}{\partial \vec{r}} + \dots$$

Keeping the electric field strength only to the first power, and using the definition (10), from (B.2) we obtain

$$W_a^2 = w^2 + 2\vec{w} \cdot \int_{t_0}^t dt' \frac{\partial}{\partial \vec{r}} \left(\vec{w} \cdot \frac{z_a e}{m_a} \vec{\varepsilon}(\vec{r}, t) \right) - \quad (B.3)$$

$$- 2\vec{w} \cdot \int_{t_0}^t dt' \int_{t'}^t dt'' \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \frac{\partial}{\partial \vec{r}} \left(\vec{w} \cdot \frac{z_a e}{m_a} \vec{\varepsilon}(\vec{r}, t) \right) +$$

$$+ \vec{w} \cdot \int_{t_0}^t dt' \left(\int_{t'}^t dt'' \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \right)^2 \cdot \frac{\partial}{\partial \vec{r}} \left(\vec{w} \cdot \frac{z_a e}{m_a} \vec{\varepsilon}(\vec{r}, t) \right) + \dots$$

Now we can insert Eq.(B.3) into the expression (38), and after expanding the exponential function, we get Eqs. (39) and (40).

Appendix C

To simplify the expression for the spatial multiplier, we can repeat step by step the calculations performed in Appendix B. Using expressions (33) and (19), according to definitions (14) and (19), we obtain:

$$\vec{\Gamma}_a = \vec{r} - \int_{t_0}^t dt' \frac{\partial H(t')}{\partial \vec{w}} + \int_{t_0}^t dt' \int_{t'}^t dt'' \left\{ \frac{\partial H(t'')}{\partial \vec{w}} \frac{\partial}{\partial \vec{r}} - \frac{\partial H(t'')}{\partial \vec{r}} \frac{\partial}{\partial \vec{w}} \right\} \cdot \frac{\partial H(t')}{\partial \vec{w}} + \quad (C.1)$$

$$- \frac{1}{2} \int_{t_0}^t dt' \left(\int_{t'}^t dt'' \left\{ \frac{\partial H(t'')}{\partial \vec{w}} \frac{\partial}{\partial \vec{r}} - \frac{\partial H(t'')}{\partial \vec{r}} \frac{\partial}{\partial \vec{w}} \right\} \right)^2 \cdot \frac{\partial H(t')}{\partial \vec{w}} + \dots$$

Let us again limit ourselves to taking into account the electric field to the first power. By introducing derivatives of expression (10) into Eq. (C.1), we obtain the equation:

$$\vec{\Gamma}_a = \vec{r} - \vec{\Sigma}_a, \quad (C.2)$$

$$\vec{\Sigma}_a = \int_{t_0}^t dt' \left\{ \vec{w} + \frac{z_a e}{m_a} \vec{\varepsilon}(\vec{r}, t') \right\} - \int_{t_0}^t dt' \int_{t'}^t dt'' \left\{ \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \frac{z_a e}{m_a} \vec{\varepsilon}(\vec{r}, t) - \frac{\partial}{\partial \vec{r}} \left(\vec{w} \cdot \frac{z_a e}{m_a} \vec{\varepsilon}(\vec{r}, t) \right) \right\} +$$

$$+ \frac{1}{2} \int_{t_0}^t dt' \int_{t'}^t dt'' \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \left(\vec{w} \cdot \frac{\partial}{\partial \vec{r}} \right) \frac{z_a e}{m_a} \vec{\varepsilon}(\vec{r}, t) + \dots$$

The above expression (C.2) can be substituted into the equation (43). It is clear that terms containing to an odd power, as a result of integration over, give zero.

References

1. Braginskii SL. Transport processes in a plasma, Reviews of Plasma Physics, v.1mm (Consultant Bureau, NY, 1965).
2. Tskhakaya DD. *Phys. Plasmas*, 31, 112301 (2024).
3. Alexandrov AF, Bogdankevich LS, Ruchadze AA. *Principles of Plasma Electrodynamics* (Springer-Verlag, Berlin, 1981), pp. 78-81.
4. Stix TH. *Waves in plasmas* (Springer Verlag, NY, 1992).