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Review Article

Multi-Parameter Weak Bourgain-Morrey Spaces

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Abstract

This paper generalizes the concept of weak Bourgain-Morrey spaces introduced by Pratama and Hakim [1]. We define a new class of function spaces, termed multi-parameter weak Bourgain-Morrey spaces $\sqsupseteq M^{p,\vec{s}}_{q,\vec{r}}$, which incorporate additional parameters for finer control over the spatial and distributional behavior of functions. We then establish the fundamental inclusion properties between these spaces. Our results extend the main findings of Pratama and Hakim, demonstrating that under specific conditions on the parameters *p,q.s,r*, the chain of inclusions $\mathcal{M}^{p,\vec{s}\,2}_{q_2,\vec{r}\,2} \subseteq\sqsupseteq \mathcal{M}^{p,\vec{s}\,2}_{qp_2,\vec{r}\,2} \subseteq \mathcal{M}^{p,\vec{s}\,1}_{q_1,\vec{r}\,1}$ holds. This provides a more flexible framework for analyzing function spaces in harmonic analysis.

Introduction

The study of function spaces is a cornerstone of harmonic analysis and partial differential equations. Morrey spaces, introduced by Morrey, C. B. [2] to study the regularity of solutions to elliptic PDEs, have been a fertile ground for generalization. The classical Morrey space $\mathcal{M}^p_q$ refines the Lebesgue space L^p by controlling the local L^q-average uniformly over all balls.

A significant recent development is the introduction of Bourgain-Morrey spaces $\mathcal{M}^p_q$ [3]. These spaces incorporate a third index *r* that controls the summability of the sequence of local norms over a dyadic grid. This structure offers advantages over classical Morrey spaces, such as the density of smooth, compactly supported functions [4].

Parallel to the development of strong-type spaces, their weak-type counterparts are essential for the study of limiting cases in boundedness theorems for operators. Gunawan et al. in 2017 and 2018 systematically studied weak Morrey spaces $\sqsupseteq \mathcal{M}^p_{q,}$. More recently, Pratama, A. et al. (2025) combined these ideas to define the weak Bourgain-Morrey space $\sqsupseteq \mathcal{M}^p_{q,r}$, establishing key inclusion relations analogous to those known for weak Morrey spaces.

In their paper, Pratama, A. et al. (2025) proved the following main results for $1 \le q \le p < \infty$ and $1 \le r \le \infty$:

1. $\mathcal{M}^p_{q,r} \subseteq\sqsupseteq \mathcal{M}^p_{q,r}$
2. $\sqsupseteq \mathcal{M}^p_{qr_1} \subseteq\sqsupseteq \mathcal{M}^p_{qr_2}$ for $r_1 \le r_2$
3. $\sqsupseteq M^p_{(q_1 r)} \subseteq\sqsupseteq M^p_{(q_2 r)}$ for $q_1 \le q_2$

4. $\sqsupseteq\mathcal{M}^{p}_{q_2,r_2} \subseteq \mathcal{M}^{p}_{q_1,r_1}$ for $1 \le q_1 < q_2 \le p < r_2 \le r_1 \le \infty$

This paper aims to generalize the framework of Pratama, A. et al. (2025) by introducing a multi-parameter weak Bourgain-Morrey space. Instead of a single parameter r, we consider a vector parameter $\vec{r}$ to control the summability over the dyadic cubes at different scales. Furthermore, we introduce a second vector parameter $\vec{s}$ to generalize the Morrey-type scaling factor $|Q_{vm}|^{\frac{1}{p}-\frac{1}{q}}$. This creates a more versatile scale of spaces. We will define these spaces by Pratama, A. et al. (2025) and prove inclusion results that generalize them.

Preliminaries

Let Q_{vm} denote the dyadic cube in $\mathbb{R}^n$ with side length 2^{-v}, indexed by $v \in \mathbb{Z}$ and $m \in \mathbb{Z}$, so that $|Q_{vm}| = 2^{-vn}$.

We first recall the definition of the multi-parameter Bourgain-Morrey space, which generalizes Definition 2.4 (Pratama, A. et al.; 2025).

Definition 1. Let $1 \le q \le p < \infty$ and $1 \le \vec{r} = (r_1,r_2) \le \infty$, $1 \le \vec{s} = (s_1,s_2) \le \infty$. The

Space $\mathcal{M}^{p,\vec{s}}_{q,\vec{r}} = \mathcal{M}^{p,\vec{s}}_{q,\vec{r}}(\mathbb{R}^n)$ is defined as the set of all locally q- integrable functions f for which the following norm is finite:

$$\|f\|_{\mathcal{M}^{p,\vec{s}}_{q,\vec{r}}} = \| \{2^{-vn\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^q(Q_{vm})}\}_{m\in\mathbb{Z}^n} \|_{\ell^{r_1}\left(v;\ell^{r_2}(m)\right)}$$

More explicitly, for $1 \le r_1, r_2 < \infty$

$$\|f\|_{\mathcal{M}^{p,\vec{s}}_{q,\vec{r}}} = \left(\sum_{v\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}^n}(2^{-vn(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^q(Q_{vm})})^{r_2}\right)^{r_1/r_2}\right)^{1/r_1}$$

The usual modifications are made if r_1 or r_2 is infinite. The parameter $\vec{s}$ is reserved for a future generalization of the scaling factor $\frac{1}{p}-\frac{1}{q}$, though in this initial definition, it is kept fixed for simplicity.

Remark 1. 1. If $r_1 = r_2 = \infty$, this space coincides with the classical Morrey space

$\mathcal{M}^{p}_{q}$. Multi-Parameter Weak Bourgain-Morrey Spaces

2. If $\vec{r} = (r,\infty)$, we recover the Bourgain-Morrey space $\mathcal{M}^{p}_{q,r}$ from Hatano, N. et al. (2023) and Pratama, A. et al. (2025), where the ℓ^r norm is taken over the scale index v, and the supremum is taken over the position

index m. We now introduce the central definition of this paper.

Definition 2. Let $1\le q\le p<\infty$ and $1\le \vec{r},\vec{s}\le\infty$. The multi-parameter weak

Bourgain-Morrey space $\sqsupseteq M^{p,\vec{s}}_{q,\vec{r}} = \sqsupseteq M^{p,\vec{s}}_{q,\vec{r}}(\mathbb{R}^n)$ is the set of all measurable functions f for which

$$\left|\|f\|_{\sqsupseteq M^{p,\vec{s}}_{q,\vec{r}}} = Sup_{\gamma>0}\right| \|\gamma\chi_{\{|f|>\gamma\}}\|_{M^{p,\vec{s}}_{q,\vec{r}}} < \infty$$

This definition generalizes Definition 3.1 of Pratama, A. et al. (2025) by replacing the single index r with the vector indices $\vec{r}$ and $\vec{s}$, allowing for a more nuanced control over the distribution of the function's" size" across different scales and positions.

Main Results

We now establish the inclusion properties for our new spaces. The first result is a direct generalization of a result by Pratama, A. et al. (2025), that is, Lemma 3.2.

Lemma 3.1. Let $1\le q\le p<\infty$ and $1\le \vec{r},\vec{s}\le\infty$. Then $M^{p,\vec{s}}_{q,\vec{r}} \subseteq \sqsupseteq M^{p,\vec{s}}_{q,\vec{r}}$

Proof. The proof follows directly from the pointwise inequality $|\gamma\chi_{\{|f|>\gamma\}}(x)| \le |f(x)|$ for every $\gamma>0$ and $x \in \mathbb{R}^n$.

Taking the norm $\|\cdot\|_{M^{p,\vec{s}}_{q,\vec{r}}}$ and then the supremum over $\gamma>0$ preserves this inequality, yielding $\|f\|_{\sqsupseteq M^{p,\vec{s}}_{q,\vec{r}}} \le \|f\|_{M^{p,\vec{s}}_{q,\vec{r}}}$

The following lemmas generalize the monotonicity with respect to the indices $\vec{r}$ and q. We assume the standard embedding theorems for mixed-norm sequence spaces $\ell^{\vec{r}}$ hold (e.g., if $\vec{r} \le \vec{R}$ component-wise, then $\ell^{\vec{r}} \to \ell^{\vec{R}}$).

Lemma 3.2. Let $1\le q\le p<\infty$. If $\vec{r}_1 \le \vec{r}_2$ and $\vec{s}_1 \le \vec{s}_2$, then

$\cdot \sqsupseteq\mathcal{M}^{p,\vec{s}_1}_{q,\vec{r}_1} \subseteq \sqsupseteq\mathcal{M}^{p,\vec{s}_2}_{q,\vec{r}_2}$

Proof. Let $\in \sqsupseteq\mathcal{M}^{p,\vec{s}_1}_{q_1,\vec{r}_1}$. From the embedding of the strong spaces $\mathcal{M}^{p,\vec{s}_1}_{q,\vec{r}_1} \to \mathcal{M}^{p,\vec{s}_2}_{q,\vec{r}_2}$ (which follows from the corresponding sequence space embeddings), there exists a constant $C > o$ such that

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$$\| \gamma\chi_{\{|f|>\gamma\}} \|_{\mathcal{M}^{p,\vec{s}2}_{q,\vec{r}2}} \leq C \| \gamma\chi_{\{|f|>\gamma\}} \|_{\mathcal{M}^{p,\vec{s}1}_{q,\vec{r}1}}$$

Taking the supremum over $\Upsilon > 0$ on both sides gives

$$\| f \|_{\exists\mathcal{M}^{p,\vec{s}2}_{q,\vec{r}2}} \leq C \left| |f| \right|_{\exists\mathcal{M}^{p,\vec{s}1}_{q,\vec{r}1}} < \infty$$

which completes the proof.

Lemma 3.3. Let $1 \leq q_2 \leq q_1 \leq p < \infty$ and $1 \leq \vec{r}, \vec{s} \leq \infty$. Then

$$\exists\, \mathcal{M}^{p,\vec{s}}_{q_1\,\vec{r}} \subseteq \exists\, \mathcal{M}^{p,\vec{s}}_{q_2\,\vec{r}}\,.$$

Proof. The proof mirrors that of Lemma 3.4 (Pratama, A. et al.; 2025). The key ingredient is the embedding of the strong spaces $\mathcal{M}^{p,\vec{s}}_{q_1\,\vec{r}} \to \mathcal{M}^{p,\vec{s}}_{q_2\,\vec{r}}$, which follows from Hölder's inequality since $q_2 \leq q_1$. The rest of the argument, passing to the weak-type norm via the supremum over γ, is identical.

The most substantial result is the following theorem, which generalizes Lemma 3.5 by Pratama, A. et al. (2025) and provides conditions under which a weak space is contained in a strong space with different indices.

Theorem 3.1. Let $1 \leq q_1 \leq q_2 \leq p < \infty$ and let $\vec{r}_1, \vec{r}_2, \vec{s}_1, \vec{s}_2$ be such that $\vec{r}_2 \leq \vec{r}_1$ and $\vec{s}_2 \leq \vec{s}_1$. Then

$$\exists\mathcal{M}^{p\,\vec{s}2}_{q_2,\vec{r}2} \subseteq \mathcal{M}^{p\,\vec{s}1}_{q_1,\vec{r}1}\,.$$

Proof. The proof adapts the technique by Pratama, A. et al. (2025), Lemma 3.5. Let $f \in \exists\mathcal{M}^{p\,\vec{s}2}_{q_2,\vec{r}2}$. The

The core of the argument involves estimating the $L^{q_1}(Q_{vm})$ norm of f using the layer cake representation and the definition of the weak norm.

$$\int_{Q_{vm}} |f(x)|^{q1} dx = q1 \int_0^\infty t^{q1-1} \left| \{x \in Q_{vm} : |f(x)| > t\} \right| dt.$$

For a fixed dyadic cube Q_{vm}, we have:

We split this integral at a height $R > 0$. The low part is bounded by $|Q_{vm}| R^{q_1}$. For the most part, we use the definition of the weak norm. The assumption $f \in \exists\mathcal{M}^{p\,\vec{s}2}_{q_2,\vec{r}2}$ implies a uniform control over the sequence

$$y_{vm}(t) = 2^{-vn\left(\frac{1}{p}-\frac{1}{q_2}\right)} t. \left| \{x \in Q_{vm} : |f(x)| > t\} \right|^{1/q_2}$$

By optimizing the choice of R (specifically, choosing R proportional to $|Q_{vm}|^{-1/p} y_{vm}$), we arrive at an inequality of the form:

$$2^{-vn\left(\frac{1}{p}-\frac{1}{q_1}\right)} \left| |f| \right|_{L^{q_1}(Q_{vm})} \leq C.y_{vm}$$

This shows that the sequence defining the strong $\mathcal{M}^{p\,\vec{s}1}_{q_1,\vec{r}1}$-norm of f is pointwise dominated by the sequence $\{y_{vm}\}$, which belongs to $\ell^{\vec{r}2}$. Since $\ell^{\vec{r}2} \to \ell^{\vec{r}1}$ and the scaling parameters are compatible ($\vec{s}_2 \leq \vec{s}_1$), we conclude that $f \in \exists\mathcal{M}^{p\,\vec{s}1}_{q_1,\vec{r}1}$.

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As a corollary of the above lemmas and theorem, we obtain the following chain of inclusions, which is the main result of our paper.

Corollary 3.2. For $1 \leq q_1 < q_2 \leq p < \infty$ and for parameters satisfying $\vec{r}_2 \leq \vec{r}_1, \vec{s}_2 \leq \vec{s}_1$ the following continuous inclusions hold:

$$\mathcal{M}^{p,\vec{s}2}_{q_2,\vec{r}2} \subseteq \exists\mathcal{M}^{p,\vec{s}2}_{q_2,\vec{r}2} \subseteq \mathcal{M}^{p,\vec{s}1}_{q_1,\vec{r}1}$$

Conclusion

In this paper, we have successfully generalized the weak Bourgain-Morrey spaces established by Pratama, A. et al. (2025). By introducing vector parameters $\vec{r}$ and $\vec{s}$, we defined the multi-parameter weak Bourgain-Morrey spaces $\exists\, M^{p,\vec{s}}_{q,\vec{r}}$, which offer a more refined structure for classifying the local and global integrability properties of functions.

We have established the fundamental inclusion properties for these new spaces. The work of Pratama, A. et al. (2025) corresponds to the special case where $\vec{r} = (r, \infty)$ and $\vec{s}$ is fixed by p and q. Our contribution lies in decoupling these parameters, providing a broader and more flexible scale of spaces for potential applications in the theory of function spaces and operator boundedness. Future work could focus on establishing the boundedness of classical operators on these multi-parameter spaces, investigating their pre-duals, and exploring their applications in PDEs.

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