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Research Article

The quadratic Poisson structures and related nonassociative noncommutative Zinbiel type algebras

Abstract

There are studied algebraic properties of the quadratic Poisson brackets on nonassociative noncommutive algebras, compatible with their multiplicative structure. Their relations both with differentiations of the symmetric tensor algebras and Yang-Baxter structures on the adjacent Lie algebras are demonstrated. Special attention is payed to the quadtatic Poisson brackets of the Lie-Poisson type, the examples of the Novikov and Leibniz algebras are discused. The nonassociated structures of commutative algebras related with Novikov, Leibniz, Lie and Zinbiel algebras are studied in details.

Introduction

Quadratic poisson brackets, their compatibility and related algebraic structures

Let $(A, +, \circ \not\gtrless)$ be a finite-dimensional a nonassociative and noncommutative algebra of dimension $N = \dim A \in \mathbb{Z}_+$ over an algebraically closed field $\mathbb{K}$. To the algebra A one can naturally relate the loop algebra $\tilde{A}$ of smooth mappings $u: \mathbb{S}^1 \to A$ and endow it with a the suitably generalized natural convolution $\prec \cdot, \cdot \succ$ on $\tilde{A}^* \times \tilde{A} \to \mathbb{K}$, where $\tilde{A}^*$ is the corresponding adjoint to $\tilde{A}$ space.

First, we shall consider a general scheme of constructing nontrivial ultra-local and local [1], quadratic Poisson structures [2-7], on the loop space $\tilde{A}$, compatible with the internal multiplication in the algebra A. Namely, let $\{e_s \in A: s = \overline{1,N}\}$ be a basis of the algebra A and its dual $\{u^s \in \tilde{A}^*: s = \overline{1,N}\}$ with respect to $\prec \cdot, \cdot \succ$ on $\tilde{A}^* \times A$, that is $\prec u^j, e_i \succ := = \delta_i^j$ for all $i, j = \overline{1,N}$, and such that for any

$$u(x) = \sum_{s=\overline{1,N}} u^s(x;u) e_s \in \tilde{A}, x \in \mathbb{S}^1,$$

the quantities $u^s(x;u) := \prec u^s(x), u \succ \in \mathbb{K}$ for all $s = \overline{1,N}, x \in \mathbb{S}^1$. Denote by

$\tilde{A}^* \wedge \tilde{A}^* := \text{Skew}(\tilde{A}^* \otimes \tilde{A}^*)$ and let $\mathscr{I}^*: \tilde{A}^* \wedge \tilde{A}^* \to \text{Symm}(\tilde{A}^* \otimes \tilde{A}^*)$ be a skew-symmetric bilinear mapping. Then for linear on $\tilde{A}$ functions $a(u) := \prec a, u \succ$ and $b(u) := \prec b, u \succ$, defined by elements $a, b \in \tilde{A}^*$, the expression

$$\{a(u), b(u)\} := \prec \mathscr{I}^*(a \wedge b), u \otimes u \succ \tag{1.1}$$

defines an ultra-local *quadratic* skew-symmetric pre-Poisson bracket on $\tilde{A}^*$. Since the algebra $\tilde{A}$ possesses its internal multiplicative structure "$\circ$", the important problem [3,4], arises: *Under what conditions is the pre-Poisson bracket* (1.1) *Poisson and compatible with this internal structure on $\tilde{A}$?* To proceed with elucidating this question, we define a co-multiplication $\Delta: \tilde{A}^* \to \tilde{A}^* \otimes \tilde{A}^*$ on an arbitrary element $c \in \tilde{A}^*$ by means of the relationship

$$\prec \Delta c, (w \otimes v) \succ := \prec c, w \circ v \succ \tag{1.2}$$

for arbitrary $w, v \in \tilde{A}$. Note that the co-multiplication $\Delta: \tilde{A}^* \to \tilde{A}^* \otimes \tilde{A}^*$, defined this way, is a *homomorphism* of the tensor algebra $\text{T}^1(\tilde{A}^*)$ into $\text{T}^2(\tilde{A}^*)$ and the linear pre-Poisson structure $\{\cdot, \cdot\}$ (1.1) on $\tilde{A}^*$ is called *compatible* with the multiplication "$\circ$" on the algebra $\tilde{A}$, if the following invariance condition

$$\Delta\{a(u), b(u)\} = \{\Delta a(u), \Delta b(u)\} \tag{1.3}$$

holds for all $a, b \in \tilde{A}^*$ and arbitrary $u \in \tilde{A}$. Now, taking into account that multiplication in the algebra A can be represented for any $i, j = \overline{1,N}$ by means of the relationship

$$e_i \circ e_j := \sum_{s=1,N} \sigma_{ij}^s e_s, \tag{1.4}$$

where the quantities $\sigma_{ij}^{s} \in \mathbb{K}$ for all i,j and $k = \overline{1,N}$ are constants, the co-multiplication $\Delta : A^* \to A^* \otimes A^*$ acts on the basic functionals $u^s \in \tilde{A}^*$ $s = \overline{1,N}$, as

$$\Delta(u^s) = \sum_{i,j=\overline{1,N}} \sigma_{ij}^s u^i \otimes u^j. \qquad (1.5)$$

Additionally, if the mapping $\vartheta^* : \tilde{A}^* \wedge \tilde{A}^* \to \text{Symm}(\tilde{A}^* \otimes \tilde{A}^*)$ is given, for instance, in the simple linear form

$$\vartheta^* : (u^i \otimes u^j - u^j \otimes u^i) \to \sum_{s,k=\overline{1,N}} (c_{sk}^{ij} - c_{ks}^{ji}) u^s \otimes u^k. \qquad (1.6)$$

The quantities $c_{sk}^{ij} \in \mathbb{K}$ are constant for all i,j and $s,k = \overline{1,N}$ and chosen to be symmetric in their below indices. Then for the adjoint to (1.6) mapping $\vartheta : \text{Symm}(\tilde{A} \otimes \tilde{A}) \to \tilde{A} \wedge \tilde{A}$ one obtains the expression

$$\vartheta : (e_s \otimes e_k + e_k \otimes e_s) \to \sum_{i,j=\overline{1,N}} c_{sk}^{ij} \, e_i \wedge e_j. \qquad (1.7)$$

Recall that a linear mapping $D : A \to B$ from an algebra A to the A-bimodule B is called a *derivation*, if for any $\lambda, \mu \in A$ there holds the Leibniz property:

$$D(\lambda \cdot \mu) = D(\lambda)\mu + \lambda D(\mu). \qquad (1.8)$$

The following theorem [3], gives an effective compatibility criterion for the multiplication in the algebra A.

Theorem 1.1: *The pre-Poisson bracket (1.2) is compatible with the multiplication (1.4) if and only if the mapping $\vartheta : \text{Symm}(\tilde{A} \otimes \tilde{A}) \to \tilde{A} \wedge \tilde{A}$ is a differentiation of the symmetric algebra $\text{Symm}(\tilde{A} \otimes \tilde{A})$.*

Proof. The idea of a proof consists in checking the relationships on the corresponding coefficients following both from the equality (1.2) and from equality

$$\vartheta(\lambda \cdot \mu) = \vartheta(\lambda)\mu + \lambda \vartheta(\mu) \qquad (1.9)$$

for basis elements $\lambda, \mu \in \text{Symm}(\tilde{A} \otimes \tilde{A})$.

Observe now that the pre-Poisson bracket (1.1) can be equivalently rewritten as

$$\prec a \wedge b, \{u \otimes u\} \succ = \prec a \wedge b, \vartheta(u \otimes u) \succ, \qquad (1.10)$$

giving rise, owing to the arbitrariness of elements $a,b \in \tilde{A}^*$, to the following tensor equality:

$$\{u \otimes u\} = \vartheta(u \otimes u) \qquad (1.11)$$

with the derivation (1.9). As was remarked in [3,4], the following natural commutator expression,

$$\vartheta(\lambda) := [r, \lambda] \qquad (1.12)$$

for any $\lambda \in \text{Symm}(\tilde{A} \otimes \tilde{A})$ and a fixed skew-symmetric constant tensor $r \in \tilde{A} \otimes \tilde{A}$ is an inner derivation of the algebra $\text{Symm}(\tilde{A} \otimes \tilde{A})$. Thus, one can consider a class of pre-Poisson brackets (1.11) in the following commutator tensor form:

$$\{u \otimes u\} = [r, u \otimes u] \qquad (1.13)$$

and pose a problem of finding conditions on the tensor $r \in \tilde{A} \otimes \tilde{A}$

under which the pre-Poisson bracket (1.13) becomes a Poisson one.

If the algebra $\tilde{A}$ is noncomutive and associative, the adjacent Lie algebra $\mathcal{L}_{\tilde{A}} \simeq \tilde{A}$ makes it possible to construct the related formal Lie group $G_{\tilde{A}} := 1 + \tilde{A}$, whose tangent space at the unity can be identified with the Lie algebra $\mathcal{L}_{\tilde{A}}$ of the right-invariant vector fields on $G_{\tilde{A}}$. For a fixed element $u \in G_{\tilde{A}}$ one can denote by $\rho_u, \xi_u : \mathcal{L}_{\tilde{A}} \to T_u(G_{\tilde{A}})$ the differentials of the right and left shifts on $G_{\tilde{A}}$, respectively. Let $\rho_u^*, \xi_u^* : T_u^*(G_{\tilde{A}}) \to \mathcal{L}_{\tilde{A}}^*$ be, respectively, dual mappings. Then, the following theorem, stated in [7], holds.

Theorem 1.2: *The following bracket*

$$\{a(u), b(u)\} = \prec \rho_u^*(b), \mathcal{R}(\rho_u^*(a)) \succ - \prec \xi_u^*(b), \mathcal{R}(\xi_u^*(a)) \succ \qquad (1.14)$$

for any $a,b \in T_u^*(G_{\tilde{A}})$ is Poisson, if the homomorphism $\mathcal{R} : \tilde{A} \to \tilde{A}$, naturally related with the tensor $r \in \tilde{A} \wedge \tilde{A}$, is skew-symmetric and satisfies the modified Yang-Baxter relationship:

$$\mathcal{R}([\alpha, \mathcal{R}\beta] + [\mathcal{R}\alpha, \beta])[\mathcal{R}\alpha, \mathcal{R}\beta] + [\alpha, \beta] \qquad (1.15)$$

for all $\alpha, \beta \in \mathcal{L}_{\tilde{A}}$ subject to the Lie commutator structure in $\mathcal{L}_{\tilde{A}}$.

If to take into account that in this case there hold the expressions

$$\rho_u^*(c) = \Delta_2 c(u), \quad \xi_u^*(c) = \Delta_1 c(u) \qquad (1.16)$$

for any $c \in \tilde{A}^*$, where the mappings Δ_1 and Δ_2 mean the convolutions of the co-multiplication $\Delta : \tilde{A} \to \tilde{A} \otimes \tilde{A}$ with the first and the second tensor components, respectively, that is

$$\prec \Delta c, u \otimes \alpha \succ = \prec c, u \circ \alpha \succ := \prec \Delta_1 c(u), \alpha \succ,$$

$$\prec \Delta c, \alpha \otimes u \succ = \prec c, \alpha \circ u \succ := \prec \Delta_2 c(u), \alpha \succ \qquad (1.17)$$

for any $\alpha \in \tilde{A}$, the bracket (1.14) will become

$$\{a(u), b(u)\} = \prec b, \mathcal{R}(\Delta_2 a(u)) \circ u \succ - \prec b, u \circ \mathcal{R}(\Delta_2 a(u)) \succ \qquad (1.18)$$

for any $a,b \in T_u^*(G_{\tilde{A}})$, which can be easily enough computed, if to take into account the relationship (1.5).

The following result [5,7], is a simple consequence of Theorem 1.2 in the case of the matrix associative algebra $\tilde{A}$ and is almost classical.

Theorem 1.3: *Let the algebra $\tilde{A}$ be matrix associative with respect to the standard multiplication, and endowed both with the natural commutator Lie structure $[\cdot,\cdot]$ and with the trace-type symmetric scalar product $\langle \cdot, \cdot \rangle := \text{Tr}(\cdot \cdot)$. Define also for the tensor*

$$r := \sum_{i,j=\overline{1,N}} r^{ij} e_i \otimes e_j \in \tilde{A} \otimes \tilde{A},$$

the related $\mathcal{R}$-homomorphism

$$\mathcal{R}\alpha := \sum_{i,j=\overline{1,N}} r^{ij} e_i < e_j, \alpha > \qquad (1.19)$$

for any $\alpha \in \tilde{A}$. Then the pre-Poisson bracket (1.18) is Poisson, if the $\mathcal{R}$-homomorphism (1.19) is skew-symmetric and satisfies the modified Yang-Baxter relationship (1.15). Moreover, the Poisson bracket (1.18) can be equivalently rewritten in the following simplified form:

$$\{a(u), b(u)\} = \prec ub, \mathcal{R}(ua) \succ - \prec bu, \mathcal{R}(au) \succ \qquad (1.20)$$

for any $a, b \in \tilde{A}^*$.

Remark 1.4: *The Yang-Baxter relationship (1.15) is basic for finding the corresponding internal multiplication structure of the algebra $\tilde{A}$, allowing the the quadratic Poisson bracket (1.18). If for example, to assume that the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ allows splitting into two subalgebras, $\mathcal{L}_{\tilde{A}} = \mathcal{L}_{\tilde{A}}^+ \oplus \mathcal{L}_{\tilde{A}}^-$, then the homomorphism $\mathcal{R} = \mathcal{P}_+ - \mathcal{P}_-$ solves the relationship (1.15), where, by definition, the mappings $\mathcal{P}_\pm : \mathcal{L}_{\tilde{A}} \to \mathcal{L}_{\tilde{A}}^\pm \subset \mathcal{L}_{\tilde{A}}$ are the suitable projections. If to assume, that the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ is generated by the associative multiplication "$*$" of the Balinsky-Novikov loop algebra $\tilde{A}$, then the related Lie structure is given by the commutator*

$$[\alpha, \beta] := \alpha * \beta - \beta * \alpha \qquad (1.21)$$

for any $\alpha, \beta \in \tilde{A}$, giving rise to the ultra-local quadratic Poisson bracket (1.18). To the regret, we do not know whether the Lie structure

$$[\alpha, \beta] := \alpha \circ D_\chi \beta - \beta \circ D_\chi \alpha \qquad (1.22)$$

for any $\alpha, \beta \in \tilde{A}$ and all $x \in \mathbb{S}^1$, suitably determining the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$, can be generated by some associative multiplication on the loop Balinsky-Novikov algebra, with respect to which the Lie structure (1.22) could entail the local quadratic Poisson bracket (1.18).

Problem 1.5: *Concerning the algebraic structures discussed above the interesting problem arises - to classify associative Balinsky-Novikov loop algebras $\tilde{A}$, whose adjacent Lie algebras $\mathcal{L}_{\tilde{A}}$ allow splitting into two nontrivial subalgebras subject to the Lie structure (1.21).*

Remark 1.6: *In the case of the basic Leibniz loop algebra $\tilde{A}$, it is well known that the usual commutator structure (1.21) does not generate the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$, yet the following inverse-derivative Lie structure*

$$[\alpha, \beta] := \alpha \circ D_\chi^{-1} \beta - \beta \circ D_\chi^{-1} \alpha, \qquad (1.23)$$

suitably determined for any $\alpha, \beta \in \tilde{A}$ and all $x \in \mathbb{S}^1$, already does the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$. Yet, we do not know whether the Lie structure (1.23) can be generated by some associative multiplication "$*$" on the loop Leibniz algebra $\tilde{A}$.

Quadratic Poisson Structures

The lie-poisson type generalization

Assume as above that $(A, +, \circ \not\sim)$ is a finite dimensional algebra of the dimension $N = \dim A \in \mathbb{Z}_+$ (in general nonassociative and noncommutative) over an algebraically closed field $\mathbb{K}$. Based on the algebra A one can construct the related loop algebra $\tilde{A}$ of smooth mappings $u : \mathbb{S}^1 \to A$ and endow it with the suitably generalized natural convolution $\prec \cdot, \cdot \succ$ on $\tilde{A}^* \times \tilde{A} \to \mathbb{K}$, where $\tilde{A}^*$ is the corresponding adjoint to $\tilde{A}$ space.

First, we will consider a general scheme of constructing nontrivial ultra-local and local [1], Poisson structures on the adjoint space $\tilde{A}^*$, compatible with the internal multiplication in the loop algebra $\tilde{A}$. Consider a basis $\{e_s \in A : s = \overline{1, N}\}$ of the algebra A and its dual $\{e^s \in A^* : s = \overline{1, N}\}$ with respect to the natural convolution $\prec \cdot, \cdot \succ$ on $A^* \times A$, that is $\prec e^j, e_i \succ := \delta_i^j$ for all $i, j = \overline{1, N}$, and such that for any

$$u(x) = \sum_{s=1, N} u_s(x) e^s \in \tilde{A}^*, x \in \mathbb{S}^1,$$

the quantities $u_s(x) := \prec u(x), e_s \succ \in \mathbb{K}$ for all $s = \overline{1, N}, x \in \mathbb{S}^1$. Denote by $\tilde{A} \wedge \tilde{A} := \text{Skew}(\tilde{A} \otimes \tilde{A})$ and let $\vartheta^* : \tilde{A} \wedge \tilde{A} \to \tilde{A} := \text{Symm}(\tilde{A})$ be a skew-symmetric bilinear mapping. Then the expression

$$\{u(a), u(b)\} := \prec u(x), \vartheta^*(a \wedge b) \succ \qquad (2.1)$$

defines for any $a, b \in \tilde{A}$ an ultra-local *linear* skew-symmetric pre-Poisson bracket on $\tilde{A}^*$. If the mapping $\vartheta^* : \tilde{A} \wedge \tilde{A} \to \text{Symm}(\tilde{A})$ is given, for instance, in the simple linear form

$$\vartheta^* : (e_i \otimes e_j - e_j \otimes e_i) \to \sum_{s=1, N} (c_{ij}^s - c_{ji}^s) e_s, \qquad (2.2)$$

where quantities $c_{ij}^s \in \mathbb{K}$ are constant for all i, j and $s = \overline{1, N}$, then for the adjoint to (2.2) mapping $\vartheta : \tilde{A}^* \to \tilde{A}^* \wedge \tilde{A}^*$ one obtains the expression

$$\vartheta : e^s \to \sum_{i, j = 1, N} (c_{ij}^s - c_{ji}^s) e^i \otimes e^j. \qquad (2.3)$$

For the pre-Poisson bracket to be a Poisson bracket on $\tilde{A}^*$, it should satisfy additionally the Jacobi identity. To find the corresponding additional constraints on the internal multiplication "$\circ$" on the algebra $\tilde{A}$, define for any $u(x) \in \tilde{A}^*, x \in \mathbb{S}^1$, the skew-symmetric linear mapping

$$\vartheta(u) : \tilde{A} \to \tilde{A}^*, \qquad (2.4)$$

called [8], by the *Hamiltonian operator*, via the identity

$$\prec \vartheta(u) a, b \succ := \prec \vartheta \cdot u, a \wedge b \succ \qquad (2.5)$$

for any $a, b \in \tilde{A}$, where the mapping $\vartheta : \tilde{A}^* \to \tilde{A}^* \wedge \tilde{A}^*$ is determined by the expression (2.3), being adjoint to it. Then it is well known [8], that the pre-Poisson bracket (2.1) is a Poisson one iff the Hamiltonian operator (2.4) satisfies the Schouten-Nijenhuis condition:

$$[[\vartheta(u), \vartheta(u)]] = 0 \qquad (2.6)$$

for any $u(x) \in \tilde{A}^*$.

Having observed that the following action

$$\vartheta(u) e_i = \sum_{s, k = 1, N} (c_{ik}^s - c_{ki}^s) u_s(x) e^k \qquad (2.7)$$

holds for any basis element $e_i \in A, i = \overline{1, N}$, the resulting pre-Poisson bracket (2.1) becomes equal to

$$\{u(a), u(b)\} = \prec \vartheta(u) a, b \succ =$$

$$= \sum_{s=1, N} \sum_{i, j = 1, N} (c_{ij}^s - c_{ji}^s) a^i b^j u_s(x) = \prec u(x), \sum_{i, j = 1, N} (c_{ij}^s - c_{ji}^s) a^i b^j e_s \succ \qquad (2.8)$$

for any $u(x) \in \tilde{A}^*$ and all $a, b \in \tilde{A}$. If now to define on the algebra A the natural adjacent to the algebra A Lie algebra structure

$$[e_i, e_j] = e_i \circ e_j - e_j \circ e_i := \sum_{s=1, N} (c_{ij}^s - c_{ji}^s) e_s \qquad (2.9)$$

for any basis elements $e_i, e_j \in A, i, j = \overline{1,N}$, the expression (2.8) yields for all $a, b \in \tilde{A}$ the well known [9,10], *classical Lie-Poisson bracket*

$$\{u(a), u(b)\} = \prec u, [a, b] \succ . \qquad (2.10)$$

Concerning the adjacent Lie algebra structure condition (2.9), it can be easily rewritten as the set of relationships,

$$\sigma_{ij}^s - \sigma_{ji}^s = c_{ij}^s - c_{ji}^s \qquad (2.11)$$

whose evident solution is

$$c_{ij}^s = \sigma_{ij}^s \qquad (2.12)$$

for any $i, j, s = \overline{1,N}$. As the bracket (2.10) is of the classical Lie-Poisson type, for the Hamiltonian operator (2.7) to satisfy the Schouten-Nijenhuis condition (2.6) is enough to check only the weak Jacobi identity for the loop Lie algebra $\mathcal{L}_{\tilde{A}}$, adjacent to the algebra $\tilde{A}$ via imposing the Lie structure (2.9), taking into account the relationships (2.12). For instance, if the commutator of the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ is given by the expression

$$[a, b] = a \circ D_\chi b - b \circ D_\chi a, \qquad (2.13)$$

the corresponding algebra A coincides with the well known Balinsky-Novikov algebra, determined by means of the following relationships

$$[a, b] = a \circ D_\chi b - b \circ D_\chi a, \qquad (2.14)$$

where, by definition, $R_a b = b \circ a = L_b a$ for any $a, b \in \tilde{A}$. If, for instance, the commutator of the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ is given by the expression

$$[a, b] = a \circ D_\chi^{-1} b - b \circ D_\chi^{-1} a \qquad (2.15)$$

for a suitably determined the inverse-differentiation mapping $D_\chi^{-1} : \tilde{A} \to \tilde{A}$, the corresponding algebra A coincides with the well known right Leibniz algebra, described by the relationships

$$[R_a, R_b] = R_{a \circ b}, \quad [R_a, R_b] = 0 \qquad (2.16)$$

for any $a, b \in \tilde{A}$. As a consequence of reasonings above one can formulate the following generalizing theorem.

Theorem: *The linear pre-Poisson bracket* (2.10) *on* $\tilde{A}^*$ *is Lie-Poisson on the adjoint space* $\mathcal{L}_{\tilde{A}}^*$ *iff the internal multiplicative structure of the algebra* A *is compatible with the weak Lie algebra structure on the adjacent loop Lie algebra* $\mathcal{L}_{\tilde{A}}$.

The same way one can consider a simple ultra-local quadratic pre-Poisson bracket on $\tilde{A}^*$ in the form

$$\{u(a), u(b)\} := \prec u(x) \otimes u(x), \vartheta^*(a \wedge b) \succ \qquad (2.17)$$

for any $a, b \in \tilde{A}$, where the skew-symmetric mapping $\vartheta^* : \tilde{A} \wedge \tilde{A} \to \mathrm{Symm}(\tilde{A} \otimes \tilde{A})$ is given for any $i, j = \overline{1,N}$ in the quadratic form

$$\vartheta^*(e_i \otimes e_j - e_j \otimes e_i) := \sum_{k,s=\overline{1,N}} (c_{ij}^{ks} - c_{ji}^{ks})(e_k \otimes e_s + e_s \otimes e_k). \qquad (2.18)$$

In particular, if to assume that the coefficients $c_{ij}^{ks} = \sigma_{ij}^k \alpha^s$ for some constant numbers σ_{ij}^k and $\alpha^s \in \mathbb{K}$ for all i, j and $k, s = \overline{1,N}$, where, by definition, the multiplications

$$e_k \circ e_s := \sum_{k=1,N} \sigma_{ij}^k e_k \qquad (2.19)$$

coincide with those of the algebra A, then the pre-Poissson bracket (2.17) yields for any $a, b \in A$ a very compact form

$$\{u(a), u(b)\} := \prec u(x) \otimes u(x), \alpha \otimes a, b] + [a, b] \otimes \alpha \succ, \qquad (2.20)$$

generalizing the classical Lie-Poisson expression (2.10) and parametrically depending on the constant vector

$$\alpha := \sum_{s=\overline{1,N}} \alpha^s e_s \in A.$$

Thus, for the pre-Lie-Poisson bracket (2.20) one can formulate suitable constraints on the algebraic structure of A. For instance, if the weak algebraic structure on the adjacent Lie algebra $\mathcal{L}_{\tilde{A}}$ is given, respectively, either by the Lie commutator (2.13) or by (2.15), then the corresponding multiplicative structures of the algebra A are generated, respectively, by the Balinsky-Novikov (2.14) and Leibniz (2.16) algebras relationships, augmented with the following common tensor multiplicative constraint

$$R_a \otimes R_\alpha = 0 = R_\alpha \otimes R_a, \qquad (2.21)$$

which holds for any $a \in A$ and a fixed element $\alpha \in A$. So, one can formulate the following theorem.

Theorem 2.2: *The quadratic pre-Lie-Poisson bracket (2.20) on* $\tilde{A}^*$ *is Poisson iff the internal multiplicative structure of the algebra A is compatible both with the weak Lie algebra structure on the adjacent loop Lie algebra* $\mathcal{L}_{\tilde{A}}$ *and with the tensor multiplicative relationships (2.21).*

In these cases there arises an interesting problem of describing the Balinsky-Novikov and Leibniz algebras, whose multiplicative structures additionally satisfy the tensor relationships (2.21). Such and related algebraic structure problems are planned to be studied in detail elsewhere. In the Section below we proceed to studying general algebraic structures related both with generalized Balinsky-Novikov and Leibniz algebras and so-called Zinbiel algebras, having diverse important applications in communications technology.

Balinsky–Novikov Type Algebraic Structures and Their Main Properties

Let $(A, +, \cdot)$ be an associative commutative algebra over a field $\mathbb{K}$ of any finite or infinite dimension (with the addition "$+$" and the multiplication "$\cdot$") and δ its derivation, i.e. $\delta : A \to A$ is a $\mathbb{K}$-linear map satisfying the Leibniz rule. Then

$$A^{\delta, \xi} = (A, +, *)$$

is a *Balinsky-Novikov algebra* (so-called *the* δ *-adjacent* or δ *-associated Balinsky-Novikov algebra* of A) with respect to "$*$" defined by the rule

$$a * b = a \cdot \delta(b) + \xi \cdot a \cdot b$$

(where ξ is a fixed element of A) and so

$$(a * b) * c = (a * c) * b$$

and

$$(a * b) * c - a * (b * c) = (b * a) * c - b * (a * c)$$

for all $a,b,c \in A$. Balinsky-Novikov algebras were introduced in connection with the so-called Hamiltonian operators [8] and Poisson brackets of hydrodynamic type [11]. Note here, that the term "*Balinsky-Novikov algebra*" was given by M. Osborn in [12]. Moreover,

$$A^{\delta,L} = (A,+,[-,-])$$

is a Lie algebra (so-called *the* δ *-adjacent* or δ *-associated Lie algebra* of A) with respect to the Lie bracket "$-,-]$" defined by the rule

$$[a,b] = a*b - b*a$$

for any $a,b \in A$ (see [13-15] and [16, p.~285]). A triple $(Z,+,\circ)$ is called *a Zinbiel algebra* (or *a dual Leibniz algebra*) if

- $(Z,+)$ is an Abelian group,

- $(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y)$,

- $(x + y) \circ z = (x \circ z) + (y \circ z)$ and $x \circ (y + z) = (x \circ y) + (x \circ z)$

for all $x,y,z \in Z$. As a consequence,

$$(x \circ y) \circ z = (x \circ z) \circ y.$$

If

$$x \odot y = x \circ y + y \circ x$$

for all $x,y \in Z$, then $(Z,+,\odot)$ is an associative commutative algebra (so-called *the adjacent* or *associated associative algebra* Z^A of a Zinbiel algebra Z). Zinbiel algebras were introduced by J.-L. Loday in [17,18] and are very popular in the control theory (in context of "chronological" algebras (see e.g. [19-22])) and in the theory of Leibniz cohomology [23].

Let $(D,+,\Diamond)$ be a (Lie, Balinsky-Novikov, Zinbiel or associative) algebra with the derivation algebra $DerD$, $\varnothing \neq \Delta \subseteq DerD$ and $\theta \in DerD$. Then D is a Lie algebra. If I is an ideal of D and $\theta(I) \subseteq I$, then we say that I is a θ *-ideal* of D. Recall that D is called:

- Δ *-simple* if $D \Diamond D \neq 0$ and any Δ -ideal I of D is 0 or D,

- Δ *-prime* if, for any Δ -ideals B,C of D, the condition $B \Diamond C = 0$ implies that $B = 0$ or $C = 0$,

- Δ *-semisimple* if, for any Δ -ideal B of D, the condition $B \Diamond B = 0$ implies that $B = 0$.

Every Δ -prime algebra is Δ -semisimple and every Δ -simple algebra is Δ -prime. If $\Delta = \{\theta\}$ and D is a Δ -simple (respectively Δ -prime or Δ -semisimple), then we say that D is θ -simple (respectively θ -prime or θ -semisimple). Moreover, if $\Delta = \{0\}$, then a Δ -simple (respectively Δ -prime or Δ -semisimple) algebra is simple (respectively prime or semisimple).

Some interesting properties of Zinbiel algebras were obtained by A.S. Dzhumadildaev, K.M. Tulenbaev [24,25] and B.A. Omirov [26]. In particular, A.S. Dzumadil'daev [25], has proved that any finite-dimensional Zinbiel algebra over the complex numbers field is nilpotent. We prove the next result.

Theorem 3.1: *Let* Z *be a Zinbiel* $\mathbb{K}$ *-algebra and* $\varnothing \neq \Delta \subseteq DerZ$. *Then the following hold:*

- if Z^A is a Δ -simple (respectively Δ -prime or Δ -semisimple) algebra, then the Zinbiel algebra Z is the ones,

- If $char\mathbb{K} \neq 2$ and Z is a 2 -torsion-free a Δ -simple (respectively Δ -prime or Δ -semisimple) Zinbiel algebra, then Z^A is a Δ -simple (respectively Δ -prime or Δ -semisimple) associative commutative algebra.

The purpose of this paper is also to study relationships between associative commutative algebras A, their δ -associated Balinsky-Novikov algebras $A^{\delta,\xi}$ and δ -associated Lie algebras $A^{\delta,L}$. Connections between properties of an associative commutative algebra A and its δ -associated algebra $A^{\delta,L}$ have been investigated by P. Ribenboim [27], C.R. Jordan, D.A. Jordan [13,14] and A. Nowicki [15]. X. Xu [28], found some classes of infinite dimensional simple Balinsky-Novikov algebras of type $A^{\delta,\xi}$. C. Bai and D. Meng [29], have proved that, if A is a finite dimensional associative commutative algebra and $0 \neq \delta \in DerA$, then $A^{\delta,0}$ is transitive (i.e.

$$r_a : A \ni x \mapsto x * a = x \cdot \delta(a) \in A$$

is a nilpotent right transformation operator of $A^{\delta,0}$ for any $(a \in A)$ and $A^{\delta,L}$ is a solvable Lie algebra [30]. In [31, Proposition~2.8] it is proved that the Balinsky-Novikov algebra $A^{\delta,\xi}$ is simple if and only if an associative commutative ring A is δ -simple. As noted in [32], there exists a conjecture: *the Balinsky-Novikov algebras* N *can be realized as the algebras* $A^{\delta,0}$, *where* A *is a suitable associative commutative algebras, and their (compatible) linear transformation.* Recall that a binary operation $G_1 : N \times N \to N$ of a Balinsky-Novikov algebra $(N,+,*)$ is called its *linear deformation* if a family of algebras $(N,+,g_q)$, where

$$g_q(a,b) = a * b + qG_1(a,b)$$

are still Balinsky-Novikov algebras for every $q \in N$. If G_1 is commutative, then it is called *compatible*.

As noted in [32], a "good" structure theory for algebraic systems means an existence of a well-defined radical and the quotient by the radical is semisimple. Our first result in this way is the following

Theorem 3.2: *Let* A *be an associative commutative algebra with* 1, $char \mathbb{K} \neq 2$, $0 \neq \delta \in DerA$ *and* $\xi \in A$. *Then the following are equivalent:*

- A is a semisimple (respectively prime or simple) algebra,

- $A^{\delta,\xi}$ is a semisimple (respectively prime or simple) Balinsky-Novikov algebra,

- $A^{\delta,L}$ is a semisimple (respectively prime or simple) Lie algebra.

Any unexplained terminology is standard as in [18,21,33,34].

An Associative Commutative Structure of a Zinbiel Algebra

Recall that a (Zinbiel or associative) algebra $(A,+,\Diamond)$ is called *reduced* if the implication

$$a \Diamond a = 0 \Rightarrow a = 0$$

is true for any $a \in A$.

Lemma 4.1: *(see [35, Theorem~3.4]) If $(Z,+,\circ)$ is a Zinbiel algebra, then $(Z,+,\odot)$ is an associative commutative ring, where "$\odot$" is defined by the rule*

$$a \odot b = a \circ b + b \circ a$$

for any $a,b \in Z$.

An additive subgroup I of a Zinbiel algebra Z is said to be *an associative ideal* of Z if $I \odot Z \subseteq I$. It is easy to see that I is an associative ideal of Z if and only if it is an ideal of Z^A.

Lemma 4.2: *Let Z be a Zinbiel algebra, $\varnothing \neq \Delta \subseteq DerZ$ and $a \in Z$. Then the following hold:*

- $a \circ Z := \{a \circ z \mid z \in Z\}$ is a right ideal of Z,

- the right annihilator

$$rann(a \circ Z) := \{t \in Z \mid (a \circ Z) \circ t = 0\}$$

of $a \circ Z$ is an associative ideal of Z,

- if I is a right Δ-ideal of Z, then $I + (Z \circ I)$ is a Δ-ideal of Z,

- if I is a Δ-ideal of Z, then the right annihilator $rannI := \{t \in Z \mid I \circ t = 0\}$ and the annihilator $annI := \{t \in Z \mid I \circ t = 0 = t \circ I\}$ are Δ-ideals, the left annihilator $lannI := \{u \in Z \mid u \circ I = 0\}$ is a right Δ-ideal of Z,

- the associated associative algebra Z^A has the identity e if and only if $a = e \circ a + a \circ e$ for any $a \in Z$,

- if $char\mathbb{K} \neq 2$, then Z is reduced if and only if Z^A is reduced,

- If Z^A has the identity e and I is an ideal of Z such that $e \in I$, then $I = Z$,

- if $char\mathbb{K} \neq 2$ and I,J are commutative ideals of Z, then $I \circ J \subseteq annZ$,

- if K is an associative Δ-ideal of Z, then

$$S(K) := \{a \in K \mid a \circ Z \subseteq K\}$$

is a right Δ-ideal of Z,

- if I,J are Δ-ideals of Z, that $I \circ J$ is the ones,

- If Z^A has identity, then every proper ideal of a Zinbiel algebra Z is contained in its maximal ideal.

Proof. Let $z,t \in Z$.

(1) Clearly that $a \circ Z$ is a subgroup of the additive group $(Z,+)$ and

$$(a \circ z) \circ t = a \circ (z \circ t) + a \circ (t \circ z) \in a \circ Z.$$

Hence $a \circ Z$ is a right ideal of Z.

(2) If $u \in rann(a \circ Z)$, then

$$(a \circ z) \circ (t \circ u) = ((a \circ z) \circ t) \circ u - (a \circ z) \circ (u \circ t) = -(a \circ z) \circ (u \circ t)$$

what gives that

$$(a \circ z) \circ (t \odot u) = 0$$

and

$$u \odot t = t \odot u \in rann(a \circ Z).$$

(3) If $i,j \in I$, then

$$t \circ (i + z \circ j) = t \circ i + t \circ (z \circ j) = t \circ i + (t \circ z) \circ j - t \circ (j \circ z) \in (Z \circ I) + I$$

and

$$(i + z \circ j) \circ t = i \circ t + (z \circ j) \circ t = i \circ t + (z \circ t) \circ j \in (Z \circ I) + I.$$

(4) We see that

$$0 = (i \circ a) \circ t = i \circ (a \circ t) + i \circ (t \circ a) = i \circ (a \odot t)$$

for any $i \in I$, $a \in rannI$,

$$0 = i \circ t) \circ a = i \circ (t \circ a) + i \circ (a \circ t) = i \circ (t \odot a)$$

and so $t \odot a = 0$ and $a \odot t \in rannZ$. Moreover,

$$0 = d(i \circ a) = d(i) \circ a + i \circ d(a) = i \circ d(a)$$

and so $d(a) \in rannI$ for any $d \in \Delta$. If $b \in lannI$, then

$$(b \circ t) \circ i = b \circ (t \circ i) + b \circ (i \circ t) = 0$$

and $b \circ t \in lannI$.

(5) Indeed,

$$a = a \odot e = e \odot a \Leftrightarrow a = e \circ a + a \circ e.$$

(6) It follows from

$$z \circ z = 0 \Leftrightarrow z \odot z = 0.$$

(7) In fact, $z = z \circ e + e \circ z \in I$ and so $I = Z$.

(8) Assume that $i \in I$ and $j \in J$. Then

$$(i \circ z) \circ j = i \circ (z \circ j) + i \circ (j \circ z) = (z \circ j) \circ i + (j \circ z) \circ i =$$
$$= z \circ (i \circ j) + z \circ (i \circ j) + j \circ (z \circ i) + j \circ (i \circ z)$$

and from this

$$2z \circ (i \circ j) = -j \circ (z \circ i). \tag{4.1}$$

By other hand,

$$j \circ (z \circ i) = (z \circ i) \circ j = z \circ (i \circ j) + z \circ (j \circ i) = 2z \circ (i \circ j). \tag{4.2}$$

Then Eqs. (4.1) and (4.2) imply that

$$j \circ (z \circ i) = 0 = (z \circ i) \circ j = z \circ (i \circ j).$$

(9) If $u \in S(A)$, then

$$(u \circ t) \circ z = u \circ (t \circ z) + u \circ (z \circ t) = u \circ (t \odot z) \in u \circ Z \subseteq A.$$

(10) Straightforward.

(11) It follows in view of the part (7).

Lemma 4.3: *Let* A *be an associative* Δ *-ideal of a Zinbiel algebra* Z, *where* $\varnothing \neq \Delta \subseteq DerZ$.. *If* $A \odot A = 0$, *then*

$$S_0(A) = S(A) + Z \circ S(A)$$

is a Δ *-ideal of* Z *such that* $S_0(A) \circ S_0(A) = 0$.

Proof. By Lemma 4.2, (9) and (3), $S_0(A)$ is an ideal of A. Let $a,b \in S(A)$ and $z,t \in Z$. Then $0 = b \odot a = b \circ a + a \circ b$ and we have that $a \circ b = -b \circ a$. Since $(z \circ b) \circ a = z \circ (b \odot a) = 0$,

$$a \circ (z \circ b) = (a \circ z) \circ b - a \circ (b \circ z) =$$
$$= (a \circ z) \circ b + (b \circ z) \circ a = (a \circ b) \circ z + (b \circ a) \circ z = (a \odot b) \circ z = 0$$

and

$$(t \circ a) \circ (z \circ b) = t \circ (a \circ (z \circ b)) + t \circ ((z \circ b) \circ a) =$$
$$= t \circ (a \circ (z \circ b)) + t \circ (z \circ (b \odot a)) = t \circ (a \circ (z \circ b)) = 0,$$

we conclude that $S_0(A) \circ S_0(A) = 0$.

Proof of Theorem 3.1: If $\Delta \subseteq DerZ$, then $\Delta \subseteq Der(Z^A)$.

Proof for simplicity. Since every Δ -ideal of Z is a Δ -ideal of Z^A, the simplicity of Z^A implies that Z is simple.

Proof for primeness. Let Z^A be a Δ -prime algebra and I, J Δ -ideals of Z such that $I \circ J = 0$. Then I, J and $J \circ I$ are Δ -ideals of Z^A and

$$(J \circ I) \circ (J \circ I) \subseteq I \circ J = 0.$$

Since $(J \circ I) \odot (J \circ I) = 0$, we conclude that $J \circ I = 0$. But then $I \odot J = 0$ and consequently $I = 0$ or $J = 0$.

(1) Proof for semisimplicity. By analogy as in the prime case.

(2) Proof for simplicity. Let Z be a Δ -simple Zinbiel algebra and A a Δ -ideal of Z^A. Then, by Lemma 4.2 (9), $S(A)$ is a right Δ -ideal of Z and, in view of Lemma 4.2 (3), $S(A) + (Z \circ S(A)) = Z$. Since $A \ni z \odot s - s \circ z = z \circ s$ for any $z \in Z$ and $s \in S(A)$, we obtain that $A = Z$.

Proof for primeness. Let Z be a Δ -prime Zinbiel algebra and I, J be Δ -ideals of Z^A such that $I \odot J = 0$. Then $i \circ j = -j \circ i$ for any $i \in I$ and $j \in J$ and

$$(i \circ z + z \circ i) \circ j = -j \circ (i \circ z + z \circ i) = -j \circ (i \circ z) - j \circ (z \circ i) =$$
$$= -(j \circ i) \circ z + j \circ (z \circ i) - j \circ (z \circ i) = -(j \circ i) \circ z = (i \circ j) \circ z$$

and

$$(i \circ z + z \circ i) \circ j = (i \circ z) \circ j + (z \circ i) \circ j = (i \circ j) \circ z + (z \circ i) \circ j$$

for any $z \in Z$ what forces that $(z \circ j) \circ i = (z \circ i) \circ j = 0$. This means that $(Z \circ I) \circ J = 0$. By the Δ -primeness of Z, $Z \circ I = 0$ (and so $I = 0$) or $J = 0$.

Proof for semisimplicity. Assume that Z is a Δ -semisimple Zinbiel algebra and A is an associative Δ -ideal of Z such that $A \odot A = 0$. By Lemma 4.3, $S_0(A)$ is an ideal of Z such that $S_0(A) \circ S_0(A) = 0$ and so $S_0(A) = 0$. If $a \circ b \neq 0$ for some $a,b \in A$, then

$$(a \circ b) \circ z = a \circ (b \odot z) \in A \circ A \subseteq A$$

for any $z \in Z$. Hence $A \circ A \in S(A)$, a contradiction. Thus $A = 0$.

As usual

$$\delta^0 = id_A$$

is the identity map of A.

Lemma 4.4: *Let* Z *be a Zinbiel algebra and* $\varnothing \neq \Delta \subseteq DerZ$. *Then the following conditions are equivalent:*

- for any Δ -ideals I, J of Z the implication

$$I \circ J = 0 \Rightarrow I = 0 \text{ or } J = 0$$

is true (i.e. Z is Δ -prime),

- for any elements $a,b \in Z$, integers $k \geq 1$, $m_i \geq 0$ and derivations $\delta_i \in \Delta$ $(i = 1,\ldots,k)$ the implication

$$(\delta_1^{m_1} \cdots \delta_k^{m_k}(a) \circ Z) \odot b = 0 \Rightarrow a = 0 \text{ or } b = 0$$

is true.

Proof. $(1) \Rightarrow (2)$ Since Z is Δ -prime, Z^A is Δ -prime by Theorem 3.1. Assume that $a,b \in Z$ and

$$(\delta_1^{m_1} \cdots \delta_k^{m_k}(a) \circ Z) \odot b = 0. \qquad (4.3)$$

Then

$$I = \sum_{k=1}^{\infty} \sum_{\substack{m_k \geq 0 \\ \delta_1,\ldots,\delta_k \in \Delta}} \delta_1^{m_1} \cdots \delta_k^{m_k}(a) \circ Z$$

is a right Δ -ideal. Moreover, $I + (Z \circ I)$ is a Δ -ideal of Z by Lemma 4.2 (3) and

$$(i + z \circ j) \odot b = (i \odot b) + z \circ (j \odot b) = 0$$

for any $i, j \in I$ and $z \in Z$. This means that $b \in rann(I + (Z \circ I))$. Inasmuch as $rann(I + (Z \circ I))$ is a Δ -ideal of Z by Lemma 4.2 (3) and $I \circ (rann(I + (Z \circ I))) = 0$, we conclude that $I = 0$ (and then $a = 0$) or $rannI = 0$ (and so $b = 0$).

$(2) \Rightarrow (1)$ Assume that $I \circ J = 0$ for some Δ -ideals I, J of Z. Then $(J \circ I) \circ (J \circ I) = 0$ and consequently $J \circ I = 0$. This gives that Eq. (4.3) is true for any $a \in I$ and $b \in J$. Hence $a = 0$ or $b = 0$. If $b \neq 0$ for some $b \in J$, then $I = 0$.

Lemma 4.5: *Let* Z *be a Zinbiel algebra and* $\varnothing \neq \Delta \subseteq DerZ$. *Then the following conditions are equivalent:*

- for any ideal I of Z the implication

$$I \circ I = 0 \Rightarrow I = 0$$

is true (i.e. Z is Δ -semisimple),

- for any elements $a,b \in Z$, integers $k \geq 1$, $m_i \geq 0$ and

derivations $\delta_i \in \Delta$ $(i = 1, \ldots, k)$ the implication

$$(\delta_1^{m_1} \cdots \delta_k^{m_k}(a) \circ Z) \odot a = 0 \Rightarrow a = 0$$

is true.

Proof. By the same argument as in the proof of Lemma 4.4.

The Balinsky–Novikov Properties

As noted in [8,36][31, Lemma~2.3], [37, Proposition~2.4], [27], the following lemma holds.

Lemma 5.1: *If A is an associative commutative algebra, $\delta \in DerA$ and $\xi \in A$, then $A^{\delta,\xi}$ is a Balinsky-Novikov algebra.*

Lemma 5.2: *Let A be an associative commutative algebra, $\delta \in DerA$ and $b,\xi \in A$. Then we have:*

- $d \in DerA^{\delta,\xi}$ if and only if $[d,\delta](b) - d(\xi) \cdot b \in annA$,

- if $1 \in A$, then $d \in DerA^{\delta,\xi}$ if and only if $[d,\delta](b) - d(\xi) \cdot b = 0$,

- $d \in DerA^{\delta,0}$ if and only if $A \cdot [d,\delta] = 0$,

- if $1 \in A$, then $d \in DerA^{\delta,0}$ if and only if $[d,\delta] = 0$.

Proof. (1) For any $a,b \in A$ and $d \in DerA^{\delta,\xi}$ we have

$$d(a) \cdot \delta(b) + a \cdot d(\delta(b)) + d(\xi) \cdot a \cdot b + \xi \cdot d(a) \cdot b + \xi \cdot a \cdot d(b) = d(a \cdot \delta(b)) + \xi \cdot a \cdot b =$$
$$= d(a * b) = d(a) * b + a * d(b) =$$
if and only if $= d(a) \cdot \delta(b) + \xi \cdot d(a) \cdot b + a \cdot \delta(d(b)) + \xi \cdot a \cdot d(b)$

$$a \cdot [d,\delta](b) + d(\xi) \cdot a \cdot b = 0.$$

(2) – (4) The rest follows from the part (1).

Lemma 5.3: *Let δ be a surjective derivation of an associative commutative algebra A with 1. If I is a right ideal of a Balinsky-Novikov algebra $A^{\delta,\xi}$, then I is an ideal of A.*

Proof. Indeed, if $i \in I$ and $a \in A$, then

$$I \ni i * a = i \cdot \delta(a) + \xi \cdot i \cdot a$$

and therefore $i * 1 = \xi \cdot i \in I$. Since δ is surjective, we have that

$$i \cdot \delta(a) = i * a - \xi \cdot i \cdot a \in I$$

and so $i \cdot A \subseteq I$.

It is easy to see that $e * e = 0$ for any idempotent $e^2 = e \in A$.

Lemma 5.4: *Let A be an associative commutative algebra, $\delta \in DerA$ and $\xi \in A$. Then the following hold:*

- [15, Lemma~3.1] if $char\mathbb{K} \neq 2$ and U is a Lie ideal of $A^{\delta,L}$, then $[U,U] = 0$ or U contains a nonzero δ-ideal of A,

- if I is a δ-ideal of A, then I is an ideal of $A^{\delta,\xi}$,

- if K is an additive δ-group of a Balinsky-Novikov algebra $A^{\delta,\xi}$, then K contains a δ-ideal

$$I_A(K) = \{k \in K \mid k \cdot A \subseteq K\}$$

of A,

- if $1 \in A$ and B is an ideal of $A^{\delta,\xi}$, then $\xi \cdot B, \delta(B) \subseteq B$,

- if $1 \in A$ and C is a left ideal of a Balinsky-Novikov algebra $A^{\delta,0}$, then $\delta(C) \subseteq I_A(C)$,

- if I is δ-ideal of A, then I is an ideal of $A^{\delta,L}$,

- if e is an idempotent of A, then $e \in rannA^{\delta,0}$,

- the kernel $\ker \delta = \{a \in A^{\delta,0} \mid \delta(a) = 0\}$ of δ is a left ideal of $A^{\delta,0}$,

- if $\delta(a) \in a \cdot A$, then $a \cdot A$ is an ideal of $A^{\delta,L}$,

- if B is an ideal of a Balinsky-Novikov algebra $A^{\delta,\xi}$, then B is an ideal of the Lie algebra $A^{\delta,L}$,

- if S is an ideal of a Balinsky-Novikov algebra $A^{\delta,\xi}$, then

$$T_A(S) = \{s \in S \mid s * A \subseteq S\}$$

is an ideal of $A^{\delta,L}$ and $T_A(S) \subseteq S$,

- if $1 \in A$ and I is an ideal $A^{\delta,L}$, then $\delta(I) \subseteq I$ and $I \cdot A$ is a δ-ideal of A,

- if W is an ideal of a Balinsky-Novikov algebra $A^{\delta,\xi}$, then $\delta(w) - \xi \cdot w \in I_A(W)$ for any $w \in W$,

- if $char\mathbb{K} \neq 2$ and $a \cdot a = 0$, then $a \cdot A$ is a right ideal of $A^{\delta,\xi}$.

Proof. (1) For proof see [15].

(2) In fact, $i * a = i \cdot \delta(a) + \xi \cdot i \cdot a \in I$ and $a * i = a \cdot \delta(i) + \xi \cdot i \cdot a \in I$ for any $i \in I$ and $a \in A$.

(3) Assume that $k \in I_A(K)$ and $x \in A$. Then

$$(x \cdot k) \cdot A = k \cdot (x \cdot A) \subseteq k \cdot A \subseteq K$$

what implies that $I_A(K)$ is an ideal of A. Since

$$\delta(k) \cdot A + k \cdot \delta(A) = \delta(k \cdot A) \subseteq \delta(K) \subseteq K$$

and $k \cdot \delta(A) \subseteq K$, we conclude that $\delta(k) \cdot A \subseteq K$. Hence $\delta(I_A(K)) \subseteq I_A(K)$.

(4) We see that $B \ni b * 1 = b \cdot \delta(1) + \xi \cdot b = \xi \cdot b$ and $B \ni 1 * b = \delta(b) + \xi \cdot b$ for any $b \in B$. Consequently $\delta(B), \xi \cdot B \subseteq B$.

(5) For any $a \in A$ and $c \in C$ we have $C \ni a * c = a \cdot \delta(c)$ what implies that $\delta(C) \subseteq I_A(C)$.

(6) For any $i \in I$ and $a \in A$

$$[i,a] = i \cdot \delta(a) - a \cdot \delta(i) \in I$$

what yields the result.

(7) Since $\delta(e) = 0$, we have $A * e = A \cdot \delta(e) = 0$.

(8) If $u \in \ker \delta$ and $a \in A$, then $\delta(a * u) = \delta(a \cdot \delta(u)) = 0$. Hence $a * u \in \ker \delta$.

(9) For any $t,b \in A$
$$[a \cdot t, b] = (a \cdot t) * b - b * (a \cdot t) = a \cdot t \cdot \delta(b) - b \cdot \delta(a) \cdot t - b \cdot a \cdot \delta(t) =$$
$$= a \cdot [t,b] - b \cdot \delta(a) \cdot t \in a \cdot A.$$

(10) Since $B * A \subseteq B, \, A * B \subseteq B$, we deduce that $[B, A] \subseteq B$.

(11) Let $a, x \in A$ and $s \in T_A(S)$. Then

$(s * x) * a = s * (x * a) - x * (s * a) + (x * s) * a$ and therefore

$[s, x] * a = s * (x * a) - x * (s * a) \in S$. For any $i \in I$

$I \supseteq [I, A] \ni [i, 1] = i * 1 - 1 * i = i \cdot \delta(i) - 1 \cdot \delta(i) = -\delta(i).$

By the part (10), W is an ideal of the Lie algebra $A^{\delta, L}$ and so

$W \ni w * a = w \cdot \delta(a) + \xi \cdot w \cdot a$

and

$W \ni [w, a] = w * a - a * w = w \cdot \delta(a) - a \cdot \delta(w)$

for any $w \in W$ and $a \in A$. Then

$W \ni w * a - [w, a] = (\xi \cdot w + \delta(w)) \cdot a.$

Hence $\delta(w) + \xi \cdot w \in I_A(W)$.

(1) We have

$0 = \delta(a \cdot a) = 2a \cdot \delta(a), \, a * a = a \cdot \delta(a) + \xi \cdot a \cdot a = 0$

and

$(a \cdot A) * (a \cdot A) \ni a \cdot b \cdot \delta(a \cdot c) + \xi \cdot a \cdot b \cdot a \cdot c =$
$\quad = a \cdot b \cdot \delta(a) \cdot c + a \cdot b \cdot a \cdot \delta(c) = 0$

for any $b, c \in A$.

If $x \in A$, then

$l_x : A^{\delta, \xi} \ni a \mapsto x * a \in A^{\delta, \xi}$

is *a left transformation operator* of the Balinsky-Novikov algebra $A^{\delta, \xi}$.

Lemma 5.5: *Let A be an associative commutative algebra, $\delta \in DerA$ and $x, \xi \in A$. Then the following hold:*

- *if $\delta \in Z(DerA) := \{\mu \in DerA \mid \mu\theta = \theta\mu \text{ for any } \theta \in DerA\}$, then $DerA \subseteq A^{\delta, 0}$,*

- *$r_x \in DerA^{\delta, 0}$ if and only if $A * (A * x) = 0$,*

- *$l_x \in DerA^{\delta, 0}$ if and only if $(A * A) * x = 0$,*

- *$[r_a, r_b] = 0$ for any $a, b \in A^{\delta, \xi}$,*

- *$[l_a, l_b] = l_{[a, b]}$ for any $a, b \in A^{\delta, \xi}$,*

- *$L(A^{\delta, \xi}) = \{l_a \mid a \in A\}$ is a Lie algebra.*

Proof. (1) If $\delta \in Z(DerA)$, then

$d(a * b) = d(a \cdot \delta(b)) = d(a) \cdot \delta(b) + a \cdot d(\delta(b)) =$
$\quad = d(a) \cdot \delta(b) + a \cdot \delta(d(b)) = d(a) * b + a * d(b)$

for any $a, b \in A^{\delta, 0}$.

(2) If $r_x \in DerA^{\delta, 0}$, then

$a \cdot \delta(b) \cdot \delta(x) = (a * b) * x = r_x(a * b) = r_x(a) * b + a * r_x(b) =$
$\quad = a \cdot \delta(x) \cdot \delta(b) + a \cdot \delta(b) \cdot \delta(x) + a \cdot b \cdot \delta^2(x)$

and so

$a \cdot (\delta(b) \cdot \delta(x) + b \cdot \delta^2(x)) = 0.$

This is equivalent to

$a \cdot \delta(b \cdot \delta(x)) = 0.$

Hence $a * (b * x) = 0$.

By the same argument as in the part (2).

$(4) - (6)$ Evident.

V.N. Zhelyabin and A.S. Tikhov [38], asked: *is true that an associative commutative algebra $(A, +, \cdot)$ with a derivation δ is δ-simple in the usual sense if and only if its corresponding Balinsky-Novikov algebra $(A, +, *)$ is simple?*

Lemma 5.6: *Let A be an associative commutative algebra, $\delta \in DerA$ and $\xi \in A$. Then A is a δ-simple algebra if and only if $A^{\delta, \xi}$ is a simple Balinsky-Novikov algebra.*

Proof. For proof see [31, Proposition~2.8].

Corollary 5.7: *Let A be an associative commutative algebra with 1, $\delta \in DerA$ and $\xi \in A$. If A is a field, then $A^{\delta, \xi}$ is a simple Balinsky-Novikov algebra.*

In the next we need the following

Lemma 5.8: *Let A be an associative commutative δ-semisimple algebra with 1, $char\mathbb{K} \neq 2$ and $\delta \in DerA$. If I is a δ-ideal of A and $\delta^2(I) = 0$, then $\delta(I) = 0$ and $I \cdot \delta(A) = 0$.*

Proof. If $i \in I$, then

$0 = \delta^2(i \cdot i) = \delta(2i \cdot \delta(i)) = 2\delta(i) \cdot \delta(i) + 2i \cdot \delta^2(i) = 2\delta(i) \cdot \delta(i)$

and therefore $\delta(i) \cdot \delta(i) = 0$. Then $(\delta(i) \cdot A)^2 = 0$ and so $\delta(i) = 0$. Moreover,

$0 = \delta(I) = \delta(I \cdot A) = \delta(I) \cdot A + I \cdot \delta(A) = I \cdot \delta(A).$

Lemma 5.9: *Let A be an associative commutative algebra with 1, $0 \neq \delta \in DerA$ and $\xi \in A$. Then A is a δ-prime algebra if and only if $A^{\delta, \xi}$ is a prime Balinsky-Novikov algebra.*

Proof. $(\Rightarrow)$ Let I and J be ideals of $A^{\delta, \xi}$ such that

$I * J = 0.$

This means that

$i \cdot \delta(j) + \xi \cdot i \cdot j = 0$

for all $i \in I$ and $j \in J$. By Lemma 5.4 (4),

$\xi \cdot I, \delta(I) \subseteq I$ and $\xi \cdot J, \delta(J) \subseteq J.$

Moreover, $annI$ and $ann(annI)$ are δ-ideals of A, $I \subseteq ann(annI)$ and

$\xi \cdot i + \delta(j) \in annI.$ \hfill (5.1)

Assume that $I \neq 0$. Then $annI = 0$ and $\delta(j) = -\xi \cdot j$ for any

$j \in J$. As a consequence, $\xi \cdot J \cdot J = 0$. Since $J \cdot A$ is a δ-ideal of A, we conclude that $J^2 \neq 0$. Then $\xi = 0$ and, in view of (5.1),

$$\delta(J) = 0. \tag{5.2}$$

In as much as

$$(J * I) * (J * I) \subseteq I * J = 0,$$

we obtain that

$$J \cdot \delta(I) \cdot \delta(I) \cdot \delta(J) + J \cdot J \cdot \delta(I) \cdot \delta^2(I) = 0$$

and by (5.1) and (5.2),

$$J \cdot J \cdot \delta(I) \cdot \delta^2(I) = 0.$$

a) If $ann(J \cdot J) \neq 0$, then $J \cdot J \subseteq ann(ann(J \cdot J)) = 0$. Hence $(A \cdot J) \cdot (A \cdot J) = 0$ and we deduce that $J = 0$.

b) Assume that $ann(J \cdot J) = 0$. Then $(\delta(I) \cdot A) \cdot (\delta^2(I) \cdot A) = 0$ and, by Lemma 5.8, $\delta(I) = 0$. As a consequence, $I \cdot \delta(A) = 0$. This means that $\delta(A) \subseteq ann(I \cdot A)$ what forces that $\delta(A) = 0$, a contradiction.

($\Leftarrow$) Let $A^{\delta,\xi}$ be a δ-prime Balinsky-Novikov algebra. Assume that X and Y are δ-ideals of A such that $X \cdot Y = 0$. By Lemma 5.4 (2), X and Y are ideals of $A^{\delta,\xi}$ and $X * Y = 0$. Thus $X = 0$ or $Y = 0$.

Lemma 5.10: *Let A be an associative commutative algebra with 1, $0 \neq \delta \in DerA$ and $\xi \in A$. Then A is a δ-semisimple algebra if and only if $A^{\delta,\xi}$ is a semisimple Balinsky-Novikov algebra.*

Proof. By the same argument as in the proof of Lemma 5.9.

Lemma 5.11: *(see [39]) Let $(N,+,*)$ be a Balinsky-Novikov algebra. Then $Z(N)$ and $[N,N]$ are ideals of N and $Z(N) * N, N] = 0$*

Lemma 5.12: *Let A be an associative commutative algebra with 1, $char\mathbb{K} \neq 2$, $0 \neq \delta \in DerA$ and $\xi \in A$. If A is a δ-prime algebra, then $Z(A^{\delta,\xi}) = 0$.*

Proof. By Lemma 5.11, $Z(A^{\delta,\xi}) * [A^{\delta,\xi}, A^{\delta,\xi}] = 0$. If $[A^{\delta,\xi}, A^{\delta,\xi}] = 0$, then

$$a \cdot \delta(b) = b \cdot \delta(a)$$

for all $a, b \in A$. Then

$$a \cdot \delta(a) \cdot b + a \cdot a \cdot \delta(b) = a \cdot \delta(a \cdot b) = a \cdot b \cdot \delta(a).$$

This gives that $a \cdot a \cdot \delta(b) = 0$ and so $a \cdot a \in ann(\delta(A) \cdot A)$. Since $ann(\delta(A) \cdot A)$ is a δ-ideal and $\delta(A) \neq 0$, we obtain that $a \cdot a = 0$. Then $(a+b) \cdot (a+b) = 0$ for any $a, b \in A$ and $a \cdot b = -b \cdot a$. This yields that $A \cdot A = 0$, a contradiction. Consequently $[A^{\delta,\xi}, A^{\delta,\xi}] \neq 0$ and thus $Z(A^{\delta,\xi}) = 0$.

It is known [31], that not all simple Balinsky-Novikov algebras have nonzero idempotents. Let A be a commutative associative ring with 1, $0 \neq \delta \in DerA$ and $\xi \in A$. If A is a then $e \in A^{\delta,\xi}$ is an idempotent if and only if $\delta(e) = -\xi \cdot e$.

Proof. Let $e = e * e \in A^{\delta,\xi}$. Then $e(\delta(e) = -\xi \cdot e) = 0$. If $e \cdot a = 0$ for some $a \in A$, then $\delta \cdot \delta(a) \in e \cdot A$ and $\delta(e) \cdot a \in ann(e \cdot A)$. Hence $(\delta(e) \cdot a)^2 = 0$. Since $(a\_0)=a\_0 A \ ^2(e) a + a\_0 A (e) a= -e (a) a$

$$a_0 = \delta(e)a$$

Recall that a nonzero ideal S of A is called *minimal* if, for any nonzero ideal P of A, the implication

Lemma 5.13: *Let A be an associative commutative algebra with 1, $\xi \in A$ and $\delta \in DerA$. If $\delta(A) \nsubseteq P$ for any minimal δ-prime ideal P of A, then:*

- every abelian ideal I of the Lie algebra $A^{\delta,L}$ is contained in the δ-prime radical $\mathbb{P}_\delta(A)$,

- $A^{\delta,L}$ is not solvable.

Proof. (1) Let I be a nonzero abelian ideal of the Lie algebra $A^{\delta,L}$. If $I \nsubseteq \mathbb{P}_\delta(A)$, then there exists a minimal δ-prime ideal of P of A such that $I \nsubseteq P$. Obviously that

$$\Delta : A / P \ni a + P \mapsto \delta(a) + P \in A / P$$

is a nonzero derivation of the quotient algebra A / P. Since A / P is a Δ-prime algebra, then $(A / P)^{\Delta, \eta}$, where $\eta = \xi + P$, is a prime Lie algebra. Hence $(I + P) / P$ is zero, a contradiction.

(2) It follows in view of the part (1).

The Lie Properties

Lemma 6.1: *[15, Theorem~3.3] Let A be an associative commutative algebra with 1 and $0 \neq \delta \in DerA$. Then A is a δ-simple algebra if and only if $A^{\delta,\xi}$ is a simple Lie algebra.*

Proof. By the same argument as in the proof of Lemma 5.9.

Lemma 6.2: *Let A be an associative commutative algebra with 1, $char\mathbb{K} \neq 2$ and $0 \neq \delta \in DerA$. If I is an abelian Lie ideal of a semisimple Balinsky-Novikov algebra $A^{\delta,\xi}$, then $\delta(I) = 0$. If, moreover, $A^{\delta,L}$ is prime, then $I = 0$.*

Proof. a) Let I be a Lie ideal of $A^{\delta,L}$ such that

$$[I, I] = 0.$$

Then $0 = [u, v] = u \cdot \delta(v) - v \cdot \delta(u)$ for any $u, v \in I$. If $x \in A$, then

$$0 = [u, [v, x]] = u \cdot \delta([v, x]) - [v, x] \cdot \delta(u) =$$
$$= u \cdot [\delta(v), x] + u \cdot [v, \delta(x)] - [v, x] \cdot \delta(u) =$$
$$= u \cdot (\delta(v) * x - x * \delta(v)) + u \cdot (v * \delta(x) - \delta(x) * v) -$$
$$- (v * x - x * v) \cdot \delta(u) =$$
$$= u \cdot (\delta(v) \cdot \delta(x) - x \cdot \delta^2(v)0 - v \cdot \delta(x) \cdot \delta(u) + x \cdot \delta(v) \cdot \delta(u)$$
$$= x \cdot (-u \cdot \delta^2(v) + \delta(v) \cdot \delta(u)) +$$
$$+ \delta(x) \cdot (u \cdot \delta(v) - v \cdot \delta(u)) + u \cdot (v \cdot \delta^2(x) - \delta(x) \cdot \delta(v)) =$$
$$= u \cdot [v, \delta(x)].$$

This means that

$$[I, \delta(A)] \subseteq I \bigcap annI = 0 \tag{6.1}$$

because $annI$ is a δ-ideal of A. If $y \in A$, then

$$0 = [u, \delta(x \cdot y)] = [u, \delta(x) \cdot y + x \cdot \delta(y)] =$$
$$= u \cdot \delta(\delta(x) \cdot y) - u \cdot \delta(x) \cdot \delta(y) - \delta(x) \cdot y \cdot \delta(u) +$$
$$+ u \cdot \delta(x) \cdot (y) + u \cdot x \cdot \delta^2(y) - x \cdot \delta(y) \cdot \delta(u) = 2u \cdot \delta(x) \cdot \delta(y).$$

Hence $\delta(A) \cdot \delta(A) \subseteq ann I$. Then $\delta(I) \cdot \delta(I) \subseteq (ann I) \cap I = 0$. Since $ann\delta(I)$ is a δ-ideal of A and $\delta(I) \subseteq ann\delta(I)$, we conclude that $\delta(I) = 0$.

b) Now assume that $A^{\delta,\xi}$ is prime. In view of (6.1),

$$0 = [I, \delta(A)] = I \cdot \delta^2(A).$$

Then $\delta^2(A) = 0$ and, by Lemma 5.8, $\delta = 0$, a contradiction. Hence $I = 0$.

Lemma 6.3: *Let* A *be an associative commutative algebra with* $\mathbf{1}$, $char\mathbb{K} \neq \mathbf{2}$, $\mathbf{0} \neq \delta \in DerA$ *and* $\xi \in A$. *Then* $A^{\delta,\xi}$ *is a prime Balinsky-Novikov algebra if and only if* $A^{\delta,L}$ *is a prime Lie algebra.*

Proof. $(\Rightarrow)$ Assume that I and J are nonzero ideals of $A^{\delta,L}$ such that

$$[I, J] = \mathbf{0}.$$

By Lemma 6.2, $[I, I] \neq \mathbf{0}$ and $[J, J] \neq \mathbf{0}$. Then I (respectively J) contains a nonzero δ-ideal I_0 (respectively J_0) of A. Since $[I \cap J, I \cap J] = 0$, we have $I_0 \cdot J_0 \subseteq I_0 \cap J_0 \subseteq I \cap J = 0$, a contradiction. Hence $A^{\delta,L}$ is a prime Lie algebra.

$(\Leftarrow)$ Suppose that B and C are δ-ideals of A such that $B \cdot C = \mathbf{0}$. By Lemma 5.4 (6), B and C are ideals of $A^{\delta,L}$. Then $[B, C] = \mathbf{0}$ and therefore $B = \mathbf{0}$ or $C = \mathbf{0}$.

Lemma 6.4: *Let* A *be an associative commutative algebra with* $\mathbf{1}$, $char\mathbb{K} \neq \mathbf{2}$, $\mathbf{0} \neq \delta \in DerA$ *and* $\xi \in A$. *Then* $A^{\delta,\xi}$ *is a semisimple Balinsky-Novikov algebra if and only if* $A^{\delta,L}$ *is a semisimple Lie algebra.*

Proof. $(\Rightarrow)$ Let I be an ideal of the Lie algebra $A^{\delta,L}$ such that $[I, I] = \mathbf{0}$. By Lemma 6.2, $\delta(I) = \mathbf{0}$. Then

$$I \ni i, a] = i \cdot \delta(a) - a \cdot \delta(i) = i \cdot \delta(a) = i * a$$

for any $i \in I$ and $a \in A$ and

$$A * I = A \cdot \delta(I) = \mathbf{0}.$$

Hence I is an ideal of the Balinsky-Novikov algebra $A^{\delta,\xi}$. Since $I * I = \mathbf{0}$, we obtain that $I = \mathbf{0}$. Thus the Lie algebra $A^{\delta,L}$ is semisimple.

$(\Leftarrow)$ Assume that B is a δ-ideal of A such that $B^2 = \mathbf{0}$. By Lemma 5.4 (6), B is an ideal of $A^{\delta,L}$ and $[B, B] = \mathbf{0}$. Consequently $B = \mathbf{0}$. Hence $A^{\delta,\xi}$ is a semisimple Balinsky-Novikov algebra.

Proof of theorem 3.2: It follows from Lemmas 5.6, 5.9, 6.1, 6.3 and 6.4.

Conclusion

We have shown that quadratic Poisson brackets generated by nonassociative noncommutative algebras and carrying many interesting algebraic properties compatible with their multiplicative structure. Their relations to the Yang-Baxter structures on the adjacent Lie algebras proved to be instructive when studying compatible Hamiltonian operators, generating integrable dynamical systems on functional spaces. It was demonstrated the importance of the quadratic Poisson brackets of the Lie-Poisson type, there were constructed Balinsky-Novikov and Leibniz algebras and investigated their internal algebraic structures. The nonassociative structures of commutative algebras related with Balinsky-Novikov, Leibniz, Lie and Zinbiel algebras were described in details.

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