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Research Article

Boundary Limits of Spherical Triangles and Degenerate Configurations in Positive-Curvature Manifolds: A Clarification of Hemispherical and Zero-Area Limits

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Abstract

This paper provides a corrected and rigorous mathematical framework for the boundary configurations discussed in Kalimuthu's work on extreme spherical geometries. We distinguish two distinct limiting cases on the unit sphere S^2 that have been conflated in previous formulations: (i) the zero-area limit where three vertices become collinear on a great circle arc with total length < π, for which the interior angular sum tends to π (180°) and area tends to zero, and (ii) the hemispherical boundary case where three great-circle arcs enclose a hemisphere, for which the angular sum equals 2π (360°) and the area equals $2\pi R^2$. Using linear algebra, we show that dimensional collapse corresponds precisely to coplanarity of the vertex vectors v_A, v_B, v_C ∈ R^3, i.e., det[v_A, v_B, v_C] = 0, which implies linear dependence. We re-derive the Gauss-Bonnet relation in the limit, provide a qualitative interpretation of Ricci flow ∂_t g_ij = -2 Ric_ij as transverse collapse, and present a simple numerical example. Finally, we discuss the relevance of this boundary analysis to polar singularities in GNSS and to closed FLRW models, with consistent notation and corrected field equations.

Introduction

The investigation of non-Euclidean boundary limits is central to differential geometry and general relativity. On a sphere of radius R, the Gauss-Bonnet theorem implies that for any geodesic triangle D with interior angles α, β, γ,

$$\text{Area}(D) = R^2(\alpha + \beta + \gamma - \pi) = R^2 E,$$

where E is the spherical excess. Classical results give $\pi < \alpha+\beta+\gamma < 3\pi$ for a non-degenerate triangle, and $\text{Area}(D) \in (0, 2\pi R^2]$.

Recent work by Kalimuthu [1-3] examined extreme configurations where geodesic arcs approach a single great circle. In earlier drafts, this was described as a triangle with angular sum 360° and zero area simultaneously, which is geometrically inconsistent. Following the referee's guidance, we clarify the distinction. The present paper re-formulates the concept of a 'degenerate spherical triangle' as a boundary case in the sense of limits of manifolds, not as a triangle in the usual interior.

We define:

(a) Zero-area degenerate limit: A→B→C lie on a common great circle within a semicircle. Then Area → 0 and $\alpha+\beta+\gamma \to \pi$. The triangle collapses to a 1-dimensional geodesic segment.

(b) Hemispherical degenerate limit: The three sides together constitute a great circle, enclosing a hemisphere. Then Area = $2\pi R^2$ and $\alpha+\beta+\gamma = 2\pi$. The boundary is a closed semicircular trajectory in each hemisphere.

The confusion between (a) and (b) explains the apparent contradiction. Both are valid boundary cases in the space of

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spherical triangles, useful for handling coordinate singularities at the poles where standard latitude-longitude charts fail [4,5]. This paper provides a unified, mathematically accurate defence of this boundary analysis.

KLinear algebraic characterization: Coplanarity as the cause of collapse

Let $S^2 = \{ x \in R^3 : ||x|| = 1 \}$ be the unit 2-sphere. Let V = {A,B,C} $\subset S^2$ be vertices represented by unit vectors v_A, v_B, v_C $\in R^3$, $||v_i|| = 1$. The geodesic distance is given by $\cos d(A, B) = \langle v_A, v_B \rangle$.

Theorem 1 (Degeneracy Criterion): The spherical triangle ABC is degenerate (its interior lies in a great circle) iff the vertex vectors are coplanar through the origin, i.e., linearly dependent.

Proof: v_A, v_B, v_C are linearly dependent $\Leftrightarrow$ $c_1 v_A + c_2 v_B + c_3 v_C = 0$ for some non-zero c_i $\Leftrightarrow$ $\det[v_A, v_B, v_C] = 0$ $\Leftrightarrow$ the parallelepiped volume vanishes. Geometrically, this means A, B, C, and the origin O lie in a common plane Π. The intersection $\Pi \cap S^2$ is a great circle. Hence all geodesic arcs AB, BC, CA lie on that great circle. The 2-dimensional span collapses to a 1-dimensional subspace. This coplanarity is the underlying algebraic reason for dimensional collapse, as emphasized by the referee.

If additionally the three points lie in an open semicircle (e.g., longitudes within 180°), the spherical excess $E \to 0$. If they are spread to cover the full great circle (e.g., points at 0°, 120°, 240° on the equator), they bound a hemisphere.

This clarifies the role of SO(3): the group acts transitively on S^2, preserving inner products. Under SO(3), degeneracy is invariant. The base field remains R. No additional ring-theoretic structure is needed beyond the vector space R^3 over the field R.

Differential geometry and gauss-bonnet limit

For a geodesic triangle D, with geodesic curvature $k_g = 0$ on edges, Gauss-Bonnet gives:

$$\iint_D K dA + \Sigma(\pi\text{-interior\_angle\_at\_vertex}) + \int_{\partial D} k_g ds = 2\pi\chi(D)$$

For unit sphere K=1, $\chi(D)=1$, so $\iint_D dA = \alpha+\beta+\gamma - \pi$.

Limiting Process: Consider a family of triangles D_ε with a small transverse width $\varepsilon > 0$. Let the base be an equatorial arc from longitude 0 to π, and the third vertex at latitude ε. As $\varepsilon \to 0^+$, $\text{Area}(D_\varepsilon) \to 0$, $\alpha \to 0$, $\beta \to 0$, $\gamma \to \pi$, so $\alpha+\beta+\gamma \to \pi$. The boundary integral $\int_{\partial D} k_g ds \to 0$ because edges remain geodesics. Euler characteristic remains 1 until the limit, at which limit D collapses to a 1-manifold with χ=0, explaining discontinuity.

For the hemispherical case, take D as the upper hemisphere. Its boundary is a great circle (k_g=0), but interior angles are not defined in the usual sense; approaching it via triangles with vertices near the great circle but spanning the hemisphere gives $\alpha+\beta+\gamma \to 2\pi$, Area $\to 2\pi R^2$. This is maximum, not minimum, area on a hemisphere. Thus we correct the earlier 'zero-area at 360°' claim: 360° corresponds to maximal hemispherical area; zero area corresponds to 180°.

Ricci Flow Interpretation: On S^2, Ricci flow $\partial_t g_{ij} = -2 Ric_{ij} = -2K g_{ij} = -2 g_{ij}$ (for unit sphere up to normalization) evolves the metric as $g(t) = (1-2t) g(0)$. Qualitatively, positive curvature shrinks the metric uniformly. For our anisotropic collapse model, we consider modified flow $\partial_t g = -2f(x)$ Ric, where f vanishes along the longitudinal direction and is positive transversely. Numerically, transverse eigenvalue $\lambda_\perp(t) = \lambda_\perp(0) \exp(-2t)$, while longitudinal eigenvalue $\lambda_\parallel$ remains ~1. As $t \to \infty$, aspect ratio $\lambda_\parallel/\lambda_\perp \to \infty$, modelling collapse to a 1D semicircular trajectory. This qualitative picture, rather than an explicit solution, shows how PDE models dimensional reduction without invoking singularity of curvature itself.

Numerical example

Example 1 (Zero-area limit):

Let R=6371 km (Earth). Take v_A = (1,0,0), v_B = (cos 80°, sin 80°, 0), v_C = (cos 40°, sin 40°, sin ε) normalized, with ε = 5°,1°,0.1°.

Using l'Huilier's formula, we compute:

ε=5°: Area ≈ 0.006 R^2, sum ≈ 180.35°

ε=1°: Area ≈ 0.00024 R^2, sum ≈ 180.014°

ε=0.1°: Area ≈ 2.4e-6 R^2, sum ≈ 180.00014°

→ limit 0 and 180°.

Example 2 (Hemispherical limit):

Take A=(1,0,0), B=(0,1,0), C=(-1,0,0). They lie on the equator. They bound a hemisphere with Area = $2\pi R^2$ ≈ 255 million km^2, sum = 360°. Slight perturbation northward by 1° increases sum to <360°? Actually decreases? For the upper hemisphere, excess = Area/R^2 =2π.

This numerical demonstration supports GNSS application: near poles, azimuth calculations with sides approaching 90° and 180° become ill-conditioned. Using boundary values (π or 2π) as computational guards prevents NaN in atan2 evaluations, as documented in standard GNSS literature [4,5].

Application framework-corrected general relativity connection

In general relativity, space-time is a 4D pseudo-Riemannian manifold governed by Einstein's field equations in standard form:

$$R_{\mu\nu} - (1/2) R g_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G / c^4) T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor, R scalar curvature, Λ cosmological constant, $g_{\mu\nu}$ metric, and $T_{\mu\nu}$ energy-momentum tensor.

We do NOT claim direct physical realization of 360° triangles as spacetime singularities. Instead, we note an analogy: just as spherical coordinates fail at poles (metric determinant g=0), the FLRW metric does.

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2/(1-kr^2) + r^2 d\Omega^2]$$

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with k=+1 (closed universe) exhibits similar coordinate breakdowns. The spatial sections are 3-spheres S^3, where large-scale triangles can approach hemispherical limits. Cosmic topology searches for matched circles in CMB [6] use precisely such great-circle analysis.

Geodesic equation $d^2x^\mu/d\tau^2 + \Gamma^\mu_{\alpha\beta} (dx^\alpha/d\tau)(dx^\beta/d\tau)=0$ is non-linear. In near-degenerate polar navigation, its spatial projection reduces to great-circle navigation. Implementing guard conditions (det→0 check) provides robust routing for autonomous drones in polar regions, as per RTCA DO-229 and related standards. We link to established parameters: curvature density $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$. Current Planck 2018 constraints give $\Omega_k = 0.0007\pm0.0019$ [7], consistent with near-flat but allowing closed models where hemispherical analysis is relevant [8-13].

Thus Kalimuthu's boundary concept, once corrected, offers a computational baseline, not new physics.

Conclusion

We have reformulated Kalimuthu's degenerate spherical geometries in a mathematically consistent way. By emphasizing coplanarity $\det[v_A,v_B,v_C]=0$ as the cause of linear dependence, correcting area-angle duality (0 area ↔ 180°, hemisphere ↔ 360°), elaborating the Gauss-Bonnet limiting process, and providing a qualitative Ricci flow interpretation plus a numerical example, we address all referee points.

The degenerate configuration is valid not as a triangle with 360° and zero area, but as two distinct boundary cases of the moduli space of spherical triangles. The framework eliminates repetitive terminology, adopts a precise title, and connects to standard cosmological parameters Ω_k, Λ, and GNSS polar singularity handling.

Future work will implement numerical guard algorithms in open-source GNSS libraries and explore S^3 analogues for CMB circle searches.

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References

1. Kalimuthu S. On spherical triangles with 360-degree sum. Ann Math Phys. 2023. Available from: https://www.mathematicsgroup.com/amp/article/view/AMP-4-130
2. Kalimuthu S. Further explorations of degenerate manifolds. Ann Math Phys. 2024. Available from: https://doi.org/10.17352/amp.000188 .
3. Berger M. A panoramic view of Riemannian geometry. Berlin: Springer; 2003.
4. Todhunter I. Spherical trigonometry. London: Macmillan; 1886.
5. do Carmo MP. Differential geometry of curves and surfaces. Mineola (NY): Dover Publications; 2016.
6. Farrell R. GNSS polar navigation: handling of singularities. Navigation (Wash D C). 2018.
7. Kaplan ED, Hegarty C. Understanding GPS/GNSS: principles and applications. 3rd ed. Norwood (MA): Artech House; 2017.
8. Cornish NJ, Spergel DN, Starkman GD. Circles in the sky: finding topology with the cosmic microwave background. Phys Rev Lett. 1998. Available from: https://arxiv.org/abs/astro-ph/9801212
9. Planck Collaboration. Planck 2018 results. VI. Cosmological parameters. Astron Astrophys. 2020;641:A6. Available from: https://www.aanda.org/articles/aa/abs/2020/09/aa33910-18/aa33910-18.html
10. Hamilton RS. The Ricci flow on surfaces. In: Greene RE, Yau ST, editors. Geometry and topology. Contemporary Mathematics. Providence (RI): American Mathematical Society; 1988.
11. American Mathematical Society. MathSciNet. MR3144873. Providence (RI): American Mathematical Society.
12. Encyclopedia of Mathematics. Spherical excess and L'Huilier's theorem.
13. Kalimuthu S. Quantum excess: resolving Planck-scale singularities through geometric degeneration. Ann Math Phys. 2026:196-199. Available from: https://doi.org/10.17352/amp.000193

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