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Review Article

Strong Mathematical Completion of Quantum Mechanics

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Abstract

SQM (Statistical QM) is a rigorously no-hidden-variables theory, based on the Schrödinger equation for a non-relativistic case where the eigenvalues of the Hamiltonian operator are possible outcomes of total energies, and wavefunctions are corresponding eigenvectors representing probability density (epistemic functions) that the outcome of measurement (in quantum ensemble) will have a particular eigenvalue. So, Schrödinger's wavefunction can not represent a real individual particle and hence shows that SQM is an incomplete theory as noted by Einstein and de Broglie. Thus, it was necessary to provide the QM theory with missed individual particles theory IQM. The individual massive non-point-like particles in IQM are represented by the 3D rest-mass energy-density topology (like the stars), generating an autogravitational force, and with hidden variables represented by the energy-density flow inside the particle's finite volume. So, a particle in QM is represented by an ontological complex wavefunction, based on its energy density and the principle of least action.

In this paper, based on Noether's continuity equation of energy-density flow, we take into consideration the ontological version of the Schrödinger equation, where the Hamiltonian operator is substituted by a fundamental classical ontological operator derived from a particle's hidden variables. Then, we demonstrate how, by quantization of this ontological PDE of a massive particle, we derive the statistical Schrödinger equation with a physical explanation why, in this process, the hidden variables disappear. Finally, we define the *classical ensemble* of repeated measurements of a real particle with identically prepared measurement systems and derive the Born rule for it.

In this way, not only are IQM and SQM two complementary parts of QM, giving the same results as demonstrated previously, but we show that the statistical QM can be derived from the more sophisticated IQM classical theory of individual particles. That is, we obtained the strong completion of Quantum Mechanics.

Introduction to Statistical Quantum Mechanics (SQM)

Every description of an experiment on a microscopic system is essentially statistical. Typically, one experiments on a sample consisting of similar microscopic systems. In an idealized theoretical description, we view such an experiment as equivalent to performing a sequence of measurements on each (now supposedly identical) microscopic system in isolation. This generates a definite result for each experiment, and the statistics of the distribution of results are described by SQM (standard Statistical Quantum Mechanics). The abstraction of an experiment shows that SQM describes the statistical

outcome of an experiment performed on a *quantum ensemble* of identical microsystems.

Unlike General Relativity (GR), current SQM is a probabilistic theory. This means it only tells us the chance that a measurement has a certain outcome. Making an observation forces a system to choose one possible value. Thus, it was in some way reduced to an epistemological discipline, based not on an affirmative model of nature but only on the image that we can have by experimental measurements. Although it is not publicly stated too frequently, Einstein had grave doubts about the various aspects of quantum mechanics. Much of the worry has revolved around the role of the observer and the question of whether quantum mechanics is an objective theory or not. In the previous

history of development of quantum theory, we have only the initial work of Einstein and that of de Broglie oriented to an individual particle, that is, to the Individual-particle Quantum Mechanics (IQM), and not to a statistical ensemble.

In the non-relativistic case, in the standard SQM, a massive elementary particle with rest-mass m_0 is represented by a probabilistic complex wavefunction $\psi(t, \vec{r})$ which defines the probability density $\overline{\psi}\psi$, and the Schrödinger equation, based on the Hamiltonian (Schrödinger defined his famous differential equation [2])

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \quad (1)$$

which is based on the Hamiltonian (total energy)

$$H(t, \vec{r}, \vec{p}) = E_k + V_e = \sum_j \frac{p_j^2}{2m_0} + V_e(\vec{r}) = \frac{|\vec{p}|^2}{2m_0} + V_e(\vec{r}) \quad (2)$$

for the momentum vector $\vec{p}$ and time-independent potential energy V_e and 3D position vector $\vec{r} = (q_1, q_2, q_3)$, we define the corresponding Hamiltonian operator (by quantization transformation of the momentum vector

$$\vec{p} \mapsto \hat{\vec{p}} \equiv -i\hbar \nabla = \sum_{j=1}^3 -i\hbar \frac{\partial}{\partial q_j} \quad (3)$$

so that:

$$\hat{H} = \sum_{j=1}^3 \frac{-\hbar^2}{2m_0} \frac{\partial^2}{\partial q_j^2} + V_e(q_1, q_2, q_3) = \frac{-\hbar^2}{2m_0} \nabla^2 + V_e(\vec{r}) \quad (4)$$

So, given a Hermitian operator $\hat{A}$, we can use the normalized solutions ψ of the Schrödinger equation, such that the scalar (inner) product

$$\langle \psi | \psi \rangle = \int \overline{\psi} \psi dV = 1 \quad (5)$$

means that the integration of the probability density $\overline{\psi}\psi$ in whole 3D space is equal to one; the average value of this operator is defined by:

$$\hat{A}_{av} = \langle \psi | \hat{A} | \psi \rangle = \int \overline{\psi} \hat{A} \psi dV \quad (6)$$

with deterministic evolution in time of the wavefunction for any fixed 3D point $\vec{r}$

$$\psi(t) = U(t)\psi(0) \quad (7)$$

for unitary $U(t)$.

From the fact that $\psi(t, \vec{r})$ are vectors of the Hilbert space, with the vector basis derived from the eigenvalues of the Hamiltonian

operator $\hat{H}$, it is possible to make a linear composition of these vectors (linear vector space), leading to the superposition and to the introduction of the operators $\hat{a}$ for the total energy, momentum, and position.

Einstein supported his opinion concerning the incomplete representation of SQM by asking whether the ψ wavefunction describes "a real condition or a mechanical system". For this purpose, he selected a periodic system which, according to SQM, possessed discrete energy states $E_1, E_2, \dots$ (eigenvalues of the Hamiltonian operator). Now, if the system in the lowest state E_1 were perturbed during a finite time by a small force, the wavefunction can be written as $\psi = \sum_{j=1}^{\infty} c_j \psi_j$ with $|c_1| \approx 1$ and $|c_j|$, $j = 2, 3, \dots$, very small quantities. But, he argued that ψ can not describe a real condition of the system, because this should have an energy exceeding E_1 by a small amount and hence it would lie between E_1 and E_2 , which is excluded by SQM. "Our ψ -function represents rather a statistical description in which c_j represent probabilities of the individual energy values", Einstein suggested:

"The ψ -function does not in any way describe a condition which could be that of a single system; it relates rather to many systems, "to an ensemble of systems" in the sense of statistical mechanics".

Initially Schrödinger intended his wavefunction as a real physical density of an individual particle, but it was demonstrated to be wrong, and the solution was the Born interpretation as a probability density to find a particle in a given position. The Schrödinger complex wavefunction can be written by $\psi(t, \vec{r}) = \sqrt{\rho(t, \vec{r})} e^{i\phi(t, \vec{r})}$ where $e^{-i\phi(t, \vec{r})}$ is a plain-wave practically present in whole 3-D space from the fact that $\vec{r}$ is a free position vector-variable. Thus, wavefunction ψ is non-local and we obtain the positive probabilistic density ρ , current $\vec{j}$ and continuity equation,

$$\rho(t, \vec{r}) = \overline{\psi}\psi, \quad \vec{j} \equiv \frac{\hbar}{i2m_0} (\overline{\psi} \nabla \psi - (\nabla \overline{\psi}) \psi) = \rho(t, \vec{r}) \left(\frac{\hbar}{m_0} \nabla \phi(t, \vec{r}) \right)$$

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \vec{j} = 0 \quad (\text{continuity equation}) \quad (8)$$

where $\frac{\hbar}{m_0} \nabla \phi(t, \vec{r})$ although appears to play the role of vector-velocity, it does not represent the velocity at a point $(t, \vec{r})$ since simultaneous measurement of position and velocity violates the uncertainty principle. Consequently, if we assume that the Schrödinger equation can be applied to an individual particle, we obtain results that are inconsistent with physical reality. The Schrödinger wavefunction in SQM describes the properties of the ensemble.

Generally, almost all "strange" results obtained in conventional SQM are obtained when SQM methods are applied to cases that are out of its proper statistical domain. In SQM, we can not speak about the trajectory of an individual particle. Note that SQM does not preclude that individual particles (systems) have definite values for all observables, but only that within the realm of SQM, we can not create an ensemble to prove it. In fact, in the complementary part of quantum theory, provided by IQM for the individual particles (systems), we

have not such problem and all observables have definite values (computable as integrals, that is, as average values of the operators).

The statistical *ensemble interpretation* (EI) makes a distinction between these complex wavefunctions and the physical entities involved. The physical entities are, for example, the electrons (an electron occupies a finite region of space which can not, at the same time, be occupied by another massive particle), but the wavefunctions are abstract (non-physical) mathematical concepts characterising the positions of the particles probabilistically, just as the action in classical mechanics is a function characterising the classical paths of particles. The explanation of the laws underlying SQM themselves need be expected only in the IQM theory, which is more fundamental because it is not statistical but considers the individual systems and particles. Thus, by providing the completion of quantum theory by new IQM, we dissolve the "mystery" from the quantum theory, and provide the only physically founded ensemble interpretation (EI) to the SQM, and hence eliminate the Copenhagen interpretation from the quantum theory. The most prominent supporters of such an interpretation were Albert Einstein and, after him, Leslie Ballentine [4]. Albert Einstein supported EI with the following words [3]:

"The attempt to conceive the quantum-theoretical description as the complete description of the individual systems leads to unnatural theoretical interpretations, which become immediately unnecessary if one accepts the interpretation that the description refers to ensembles of systems and not to the individual systems".

The EI of SQM claims that it is *minimalist*, making the fewest assumptions about the meaning of the Schrödinger wavefunction ψ and proposes to take to the fullest extent the statistical interpretation of Max Born. The attraction of the EI is that it immediately dispenses with the metaphysical issues associated with reduction (collapse) of the state vector (wavefunction) ψ , Schrödinger cat states, and other issues related to the concepts of multi-simultaneous states. The EI postulates that the wavefunction is never physically required to be reduced (collapsed). It is clear that on each measurement, only one of the possible states will be observed, but there is no requirement for any notion of collapse of the wavefunction. The state is not taken to be physically real or to be a literal summation of states. The wavefunction is taken to be an abstract statistical (epistemic) function that people need in the presence of incomplete knowledge (but nature does not assign epistemic concepts to exist, and always obeys the physical laws), only applicable to the statistics of repeated preparation procedures for the measurements, similar to Statistical Classical Mechanics (SCM).

The ensemble in SQM is described by normalized probabilistic-density (with pure state operator $|\psi\rangle\langle\psi|$) of the complex wavefunction ψ , such that the scalar product from (5), is $\langle\psi|\psi\rangle = \int \bar{\psi}\psi dV = 1$, and the results of measurement of an observable quantity A (of a linear Hermitian operator $\hat{A}$, such that for any basis vector e_k of this Hilbert space $\hat{A}e_k = \alpha_k e_k$, where $\alpha_k \in \mathbb{R}$, and any vector $\psi = \sum c_k e_k$ with $c_k = \langle e_k | \psi \rangle \in \mathbb{C}$) for the entire ensemble can be described as follows:

1. Only eigenvalues α_k can be obtained, one of them for "each

measurement on an individual system in the ensemble". Collecting the data on the individual measurements yields the *statistical* information.

2. Born rule: Each possible value shows up with probability

$P_k = |c_k|^2 = \overline{c_k} c_k = |\langle e_k | \psi \rangle|^2$. This probability depends on the sample, described by the normalized wavefunction ψ , and it depends on the observable being measured (through its eigenfunction e_k).

3. After measuring A , the initial ensemble is split up into subensembles, one corresponding to each of the possible eigenvalues α_k . The wavefunction that describes the subensemble corresponding to is the corresponding normalized eigenstate e_k .
4. If we were to continue to pursue some further measurements on the complete ensemble, we would work with each of the subensembles described by and collect the results by summing over each subensemble and multiplying this subresult by the respective probability P_k .

So, the ensemble interpretation provides a consistent statistical framework for the SQM, which can not be applied to the deterministic classical quantum theory IQM for the individual systems (particles). On the other hand, the usual form of SQM *does not say anything about actual deterministic causes that lie behind the probabilistic quantum phenomena*. This fact is often used to claim that SQM implies that nature is fundamentally random. SQM declared the rigorous *no-hidden-variable* theorems. These theorems are often used to claim that hidden variables cannot exist and, consequently, that nature is fundamentally random. However, this assumption alone is not sufficient to provide a theorem, and such an assumption is not valid: the hidden variables (the flux of energy density of an individual massive particle) are used in IQM, provided in the next section, and they do not violate the principle of locality. Therefore, the claim that quantum theory implies fundamental randomness is only a myth.

Previous Classical to Quantum Mechanics Method: In this derivation of statistical Schrödinger equation from classical deterministic IQM theory of individual particles, we will use the operatorial formulation of this classical IQM. This method has been used by Koopman and von Neumann (KvN) in 1931-32 [7, 8], and has been based on *statistical classical mechanics of Gibbs' ensemble concept* [5], for a mechanical system with n degrees of freedom, where the elementary *phase volume* is equal to $dx \equiv dr_1 \dots dr_n dp_1 \dots dp_n = dx_1 \dots dx_{6n}$, where $\vec{r}_i$ are the generalized coordinates and $\vec{p}_i$ are the generalized momentum of the system. The phase volume of a finite phase region $V_P \subseteq M$ is equal to the $6n$ -dimensional integral $\int_{V_P} dx$. In general, an ensemble is defined by its phase space distribution function $f(t, \mathbf{x})$, for $\mathbf{x} \in M$, which may possibly depend explicitly on time t . Let ∇_x be the $6n$ -dimensional gradient on the phase space

$$\nabla_x \equiv \left(\frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right) = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{6n}} \right)$$

and the 'velocity vector' in phase space at a point x , which

can be seen as the vector of the velocity field $\mathbf{W}$ at a point $\mathbf{X}$,

$$\mathbf{w} = \mathbf{W}(\mathbf{x}) \equiv \frac{d\mathbf{x}}{dt} = \left(\frac{dx_1}{dt}, \dots, \frac{dx_{6n}}{dt} \right) = \left(\frac{d\mathbf{r}_1}{dt}, \dots, \frac{d\mathbf{r}_n}{dt}, \frac{d\mathbf{p}_1}{dt}, \dots, \frac{d\mathbf{p}_n}{dt} \right) = \left(\frac{\partial H}{\partial \mathbf{p}_1}, \dots, \frac{\partial H}{\partial \mathbf{p}_n}, -\frac{\partial H}{\partial \mathbf{r}_1}, \dots, -\frac{\partial H}{\partial \mathbf{r}_n} \right)$$

if the system is described by Hamilton's equations of motion, so that we have that this 'velocity vector' has zero divergence in Hamiltonian systems,

$$\nabla \cdot \mathbf{x} \cdot \mathbf{w} \equiv \nabla \cdot \mathbf{x} \cdot \left(\frac{d\mathbf{x}}{dt} \right) = \sum_{i=1}^n \left(\frac{\partial^2 H}{\partial \mathbf{r}_i \partial \mathbf{p}_i} - \frac{\partial^2 H}{\partial \mathbf{p}_i \partial \mathbf{r}_i} \right) = 0 \quad (9)$$

And the system's phase volume remains constant when the system moves (Liouville's theorem). In fact, the *phase space probabilistic density* function f , such that at each instance of time t , $\int_M f(t, \mathbf{x}) d\mathbf{x} = 1$, must satisfy the Liouville's equation [6] for the divergence-less flows, which is just a version of the Noether continuity equation for the conserved current¹, which, for systems governed by Hamiltonian dynamics (9), reduces to Liouville's equation (or Gibbs's "conservation of density in phase space"):

¹The probability density f in phase space can be defined as the relative number density of the phase points associated with an infinite statistical ensemble. The statistical ensemble have to be infinite because otherwise f would be a step function with an a.e. null derivative. Imposing some conditions on the statistical ensemble and using the Radon-Nikodym theorem, one can rigorously define f as a function with continuous partial derivatives over all the phase space.

$$\frac{df(t, \mathbf{X})}{dt} = \frac{\partial f(t, \mathbf{X})}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla f(t, \mathbf{X}) = 0 \quad (10)$$

That is, viewing the motion through phase space as a "fluid flow" of system points, the theorem that the convective derivative of the density, $\frac{df(t, \mathbf{x})}{dt}$, is zero follows from Noether's equation of continuity by noting that the "velocity field" $\frac{d\mathbf{x}}{dt}$ in phase space has zero divergence (which follows from Hamilton's relations).

The starting point of their work is the possibility of defining a Hilbert space of *complex* and *square integrable* classical "wave" functions $\psi^c(t, \mathbf{x})$ such that

$$f(t, \mathbf{x}) \equiv |\psi^c(t, \mathbf{x})|^2 = \psi^c(t, \mathbf{x}) \overline{\psi^c(t, \mathbf{x})} \quad (11)$$

Can be interpreted as a probability density of finding a particle at the point $(t, \mathbf{x})$ of the phase space. So, KvN proposed to extend it to a complex "classical wave function" in the phase space

$$\psi^c(t, \mathbf{x}) = \sqrt{f(t, \mathbf{x})} e^{\frac{i}{\hbar} S(t, \mathbf{x})} \quad (12)$$

with the inner product defined in the phase-space as

$$\langle \psi_1^c, \psi_2^c \rangle \equiv \int \overline{\psi_1^c} \psi_2^c d\mathbf{x}. \text{ The relations presented above represent}$$

the basic equations of the KvN theory, as developed by Gozzi, Mauro [9], and others. In the KvN formulation of Statistical Classical Mechanics (SCM), the Hilbert space is made up of complex "wave functions" over the phase space variables. It was demonstrated, recently in 2017, [10] that the proper physical meaning of the phase-space function is the classical action (see (12) in [10]) during the change of the phase-space from the initial value $\mathbf{x}_0$ (at $t = 0$) to the value $\mathbf{x}_t$ at some later time $t > 0$. So, the Liouville equation for the 'classical wave function' in the phase-space (the equation (19) in [10]) :

$$\left(\frac{\hbar}{i} \frac{\partial}{\partial t} + \sum_{i=1}^n \left[-\frac{\hbar}{i} \frac{\partial H}{\partial \mathbf{r}_i} \frac{\partial}{\partial \mathbf{p}_i} + \frac{\partial H}{\partial \mathbf{p}_i} \left(\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}_i} - \mathbf{p}_i \right) \right] + H(\mathbf{x}) \right) \psi^c(t, \mathbf{x}) = 0 \quad (13)$$

Remark: Quantization principle: The "quantization" of the classic wave equation has to substitute the "classical wave function" $\psi^c(t, \mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{p}_1, \dots, \mathbf{p}_n)$ by the quantum wavefunction $\psi(t, \mathbf{r}_1, \dots, \mathbf{r}_n)$ by elimination of all momentum components, so that $\frac{\partial}{\partial \mathbf{p}_i} \psi = 0$, and by replacing the momentum $\mathbf{p}_i$ with its quantum operator $\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}_i}$ used for quantization of the Hamiltonian used by Schrödinger in (1).

So, the classical wave equation above reduces into the following quantum equation

$$\left(\frac{\hbar}{i} \frac{\partial}{\partial t} + H(\mathbf{r}_1, \dots, \mathbf{r}_n, \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}_1}, \dots, \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}_n}) \right) \psi(t, \mathbf{r}_1, \dots, \mathbf{r}_n) = 0, \text{ which}$$

is just the Schrödinger equation. This derivation shows that the version (13) of the Liouville equation may be interpreted as classical counterpart of the Schrödinger equation. The common quantization rule, replacing

a momentum $\mathbf{p}_i$ by a derivative $-\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}_i}$, can be understood as part of a projection from phase space $(t, \mathbf{x}) = (t, \mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{p}_1, \dots, \mathbf{p}_n)$ to configuration space $(t, \mathbf{r}_1, \dots, \mathbf{r}_n)$ for n particles.

□

Thus, in next we will use this method where the phase space is replaced by 4D Minkowski timespace, the probability density $f(t, \mathbf{x})$ in phase space by the rest-mass energy-density $\Phi_m(t, \vec{\mathbf{r}})$, the classical action $S(t, \mathbf{x})$ by classical least action $S(t, \vec{\mathbf{r}}T)$ on particle's trajectory $\vec{\mathbf{r}}T(t)$, and $\psi^c(t, \mathbf{x}) = \sqrt{f(t, \mathbf{x})} e^{\frac{i}{\hbar} S(t, \mathbf{x})}$ is replaced by ontological wavefunction Ψ of an individual particle.

Short Introduction of Ontological IQM Theory of Individual Particles

The problem about an incompleteness of Quantum Mechanics was discussed from the beginning of the early development of the Quantum Mechanics based on the concepts of the point-like elementary particles. Nevertheless, even one century since its birth, many problems related to interpretation of this theory persists: non-

local effects of entangled states, wave function reduction and the concept of measurement in QM, the transition from a microscopic probabilistic world to a macroscopic deterministic world and so on. A possible way out from these problems would be if QM represents a statistical approximation (suggestion dating since the EPR (Einstein, Podolsky, Rosen) paper of 1935) of an unknown deterministic theory, where all observables have defined values fixed by 'hidden' unknown variables. The main idea of Einstein to resolve this problem has been to consider a massive elementary particle not as a point-like object. So, he proposed the idea that a real physical massive particle can be represented by the packet of energy density localized at each instance of time in a small 3D volume. My most important teacher is Albert Einstein, with this message in [15]:

"We could therefore regard matter as being constituted by the regions of space in which the field is extremely strong. A thrown stone is, from this point of view, a changing field in which the states of the greatest field intensity travel through space with the velocity of stone. There is no place in this new kind of physics both for the field and the matter, for the Field is the only reality...and the laws of motion would automatically follow from the laws of field" and Luis de Broglie [16]:

"In my view, the wave is a physical one having a very small amplitude which cannot be arbitrarily normed, and which is distinct from the Ψ wavefunction. The latter is normed and has a statistical significance in the usual quantum mechanical formalism... For me, the [individual] particle, precisely located in space at every instant, forms on the Ψ wave a small region of high energy concentration, which may be likened in a first approximation, to a moving singularity."

These principles have been accepted at a very beginning of my research [17-20].

By *completion* of Quantum Mechanics we mean that this theory should be consistent with obtained results of quantum mechanics, so that it should fully reproduce all the quantum outcomes but it could provide a more refined description of the microscopic reality. The family of completions of quantum mechanics has been usually denoted as hidden variable models, although more recently the term 'ontological models' is preferred. In an ontological model of quantum mechanics there is a deeper specification of the state of the particle, and in this approach to completion I specified these states by the energy-density distributions of a given particle in the Minkowski time-space. Such an ontic state, also not fully accessible (non fully observable by the measurements, is represent the complete description of an elementary particle (or of the system of elementary particles), in order to be able to compute from it all properties of a particle as, for example, its rest-mass, position, speed, momentum, total energy, etc... The main part of this research program to define the *ontological* wavefunctions Ψ and new equations for them has been provided by three volumes [21-23], and by my Research Gate open-access book [24].

The idea about inadequate representation of a *given individual* elementary particles is based also on the fact that the Hamiltonian operators are adequate mainly to find the set of *all* possible energy levels which possibly can have a particle in a given situation, and to associate a probability to each such a possible energy level. However, the differential wave-function equations based on the Hamiltonian are not adequate to represent the current state, properties and concrete deterministic trajectory of a given individual particle with a *given energy-momentum level*. We need a new complementary set of

quantum operators and differential equations, to describe the 'reality' of such an individual concrete particle, as a deterministic system of the time-evolution of such a concrete particle with appropriate Hilbert spaces.

The proposed completion of QM theory, with the new non-probabilistic equations and a new mathematical basis for the deterministic quantum mechanics, presented here as a conservative extension of the Standard Quantum Mechanics by non point-like elementary particles, can reshape our view of the quantum world, allowing us to include also classical gravity and to answer some of the deep unresolved questions at the heart of quantum mechanics. This new theory of particles is a constructive approach where the massive particles are not point-like but the 3D conservative distributions of energy/matter in a finite volume (for each fixed time-instance) during their propagation. Thus, it avoids the infinitary problems of the inverse square law for gravitational and electric forces, and may be used as a formal basis for the Einstein's unification theory. It is known that if the mass m of the particle is inside a sphere with Schwarzschild radius

$$r_s = \frac{2Gm}{c^2} \text{ then it would become a black hole. Hence, it is clear that}$$

the massive particles can not be a point-like particles as in SQM, and that we need the non point-like models for the massive particles in order to render valid both special and general relativity theory for the quantum mechanics.

There is no single universally agreed scientific meaning of the word *matter*. Scientifically, the term *mass* is well-defined (it is a mathematical relation between magnitude of external force acting on a matter body and the body's linear acceleration, for any kind of bodies in the time-space, thus also for the particles), but matter is not. In my approach the matter is different from the rest-mass (which is only a physical property of the matter), and that all particles are composed by matter.

Still, special relativity shows that matter may disappear by conversion into energy, even inside closed systems, and it can also be created from energy, within such systems. The quantity of energy remain the same during a transformation of matter (which represents a certain amount of energy) into non-material (i.e., non-matter) energy. This is also true in the reverse transformation of energy into matter.

In what follows, we considered the matter as a stuff from which is build any particle, but it will be used the more rigorous physical concept of its observable *rest-mass energy* property that is time-invariant in any given inertial Minkowski reference frame (it is fundamental experimental property, because each physical experiment is strictly connected to its locally-flat inertial reference frame).

Remark: Here, for the non-relativistic case, we will consider the standard 3D open space dimensions with the time dimension. So, a vector of position in this time-time 4-dimensional system is given by

the *Minkowski tetrad* (4-dimensional vector base) (e_0, e_1, e_2, e_3) and the reals $\eta_{ij} = e_i \cdot e_j$, where the matrix (η_{ij}) (and its inverse

$$(\eta^{ij}) \text{ is given by the metric tensor } (\eta_{ij}) = (\eta^{ij}) = \text{diag}(1, -1, -1, -1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (14)$$

so that

- Any point of this time-space is given by a *position four-dimensional vector* (or *Minkowski tetrad* [1])

$$r_4 = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = cte_0 + xe_1 + ye_2 + ze_3 = cte_0 + \vec{r}$$

where t is the time (i.e., ct is the time-like component of $\vec{r}_4$, where c is the velocity of light in the vacuum) and

$\vec{r} = xe_1 + ye_2 + ze_3$ is an ordinary Euclidean vector with three spatial coordinates x, y, z , and with squared Minkowski norm $\|\vec{r}_4\|^2 \equiv r_4 \cdot r_4 = (ct)^2 - x^2 - y^2 - z^2$, and $\|\vec{r}\|^2 \equiv -\vec{r} \cdot \vec{r} = x^2 + y^2 + z^2 \geq 0$.

- The 4-dimensional angular wavenumber,

$$k_4 = k_t e_0 + k_x e_1 + k_y e_2 + k_z e_3 \equiv \frac{\omega}{c} e_0 + \vec{k},$$

where $\hbar = \frac{h}{2\pi}$ is the Dirac's constant, for the Planck's constant $h = 6.6210^{-34} Js$. ω is the angular frequency and

$$\vec{k} = k_x e_1 + k_y e_2 + k_z e_3, \quad \text{with} \quad k^2 = \|\vec{k}\|^2 = k_x^2 + k_y^2 + k_z^2$$

is the spatial component of the angular wavenumber vector:

$$k_x = \frac{2\pi}{\lambda_x}, k_y = \frac{2\pi}{\lambda_y}, k_z = \frac{2\pi}{\lambda_z}, \quad \text{where } \lambda_x, \lambda_y, \lambda_z \text{ are spatial}$$

wavelengths w.r.t the axes x, y and z respectively.

De Broglie established that each 4-momentum vector of a given particle P_4 defines the angular wavenumber vector k_4 , such that $p_4 = \hbar k_4 = \frac{\hbar \omega_0}{c} e_0 + \vec{p} = \frac{E}{c} e_0 + \vec{p}$, where $\vec{p} = \hbar \vec{k}$ is the 3-dimensional momentum (obtained from derivative of Lagrangian) and $H = \hbar \omega_0$ the total particle's energy.

- $\nabla \equiv e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z}$ is the gradient, so that the

Laplacian is defined by

$$\Delta \equiv -\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}; \quad \text{and} \quad \text{divergence,}$$

$$\nabla \cdot \vec{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

□

Fundamental assumption in this theory of the elementary particles is that they can be represented in the n -dimensional pseudo-Euclidean Minkowski space, with 1+3 open time-space dimensions (14), by the complex wave-packets (the Einstein's proposed field in the form of a

generalized Fourier expression) Ψ [17], based on the fact that ω depends on $\vec{k}$,

$$\Psi(r_4) = \int C(k_4) e^{i(-k_4 r_4)} dk = \int_{\vec{k} \in \mathbb{R}^3} A(\vec{k}) e^{i(-\vec{k} \vec{r} - \omega(\vec{k})t)} dk \\ = \iiint_{-\infty}^{+\infty} A(\vec{k}) e^{i(k_x x + k_y y + k_z z - \omega(\vec{k})t)} dk_x dk_y dk_z \quad (15)$$

where $A(\vec{k}) = C(k_4)$ (because the component $k_t = \frac{\omega(\vec{k})}{c}$ is a function of $\vec{k}$), such that

Moreover, from the fact that a time-space perturbation Ψ is just a local time-space curvature (from the General Relativity point of view), only if the particle follows the time-space geodesics (that is, when propagates with a *constant velocity*) then the complex coefficients $A(\vec{k})$ in (15) are constant, that is does not depend on time-space; Otherwise, during the accelerations, the particle changes locally the time-space geometry, and the coefficients $A(\vec{k})$ become dependent both on time and space (i.e, time-space dependent).

$$\Phi_m(r_4) = |\Psi|^2 = \overline{\Psi} \Psi \quad (16)$$

is the real *rest-mass energy density* of the massive elementary particle, and by definition of the real function $\Phi = \sqrt{\Phi_m}$ and r_4 substituted by $(t, \vec{r})$, from derivation in Section 2 in [23], we obtain the *mathematical* representation of any *individual* massive object (elementary particles and all bound multi-particle material objects of Universe) by *ontological wavefunction*

$$\Psi = \Phi(t, \vec{r}) e^{-i\varphi_T} \quad (17)$$

(different from epistemological Schrödinger's wavefunctions denoted by Ψ) is only a mathematical representation of two fundamental physical phenomena that determine the movement of each material object at a given time-instant t :

- Object's rest-mass energy-density, delimited in a finite 3-D space at each instance of time t and where $\vec{r}$ is a *free* 3-D vector-variable (a space point) differently from the derived (non free) vector variable $\vec{r}_T(t)$ used for object's barycenter,

$$\Phi_m = \Psi \overline{\Psi} = \Phi^2(t, \vec{r}) \quad (18)$$

With this, by fixing that material objects in the Universe, from elementary particles to stars, are *representable by rest-mass energy-densities*, we obtain a direct connection with curved time-space General Relativity.

- The de Broglie pilot-wave phase φ_T of mathematical object's ontological wavefunction Ψ , computed in object's barycenter $\vec{r}_T(t)$, which represents the *principle of least action* (of object's Lagrangian $L(t, \vec{r}_T, \vec{v})$) for a canonical momentum $\vec{p}(t)$, speed (group velocity of particle's

$$\text{barycenter)} \quad \vec{v}(t) = \frac{d}{dt} \vec{r}_T \quad \text{and total energy (Hamiltonian } H) \\ \varphi_T(t, \vec{r}_T) \equiv -\frac{1}{\hbar} S|_{t_0=0} = \frac{1}{\hbar} \int_0^t (\vec{p}(t') \vec{v}(t') + H(t', \vec{r}_T(t'), \vec{p}(t'))) dt' \quad (19)$$

where $S|_{t_0=0}$ is the *Hamiltonian principal function* when initial position $t_0 = 0$ (the necessary condition to have $\varphi_T = 0$ for $t = 0$), which is the property of Universe valid for movement of any object in open 4D time-space.

$$\text{Thus,} \\ -\frac{1}{\hbar} L = \frac{d}{dt} \varphi_T(t, \vec{r}_T) = \frac{\partial \varphi_T}{\partial t} - \frac{\partial \varphi_T}{\partial \vec{r}_T} \frac{d\vec{r}_T}{dt} = \frac{\partial \varphi_T}{\partial t} - \frac{\partial \varphi_T}{\partial \vec{r}_T} \vec{v} \\ , \text{ where} \\ \frac{\partial \varphi_T}{\partial \vec{r}_T} = -\frac{1}{\hbar} \vec{p} \quad (20)$$

and during the observation, from $\frac{\partial}{\partial t} \varphi_T = -\frac{1}{\hbar} (L + \vec{p} \vec{v})$, we obtain in a general case (with accelerations as well) at each instance of time $t \geq 0$,

$$\omega_p \equiv \frac{\partial \varphi_T}{\partial t} = \frac{H(t, \vec{r}_T, \vec{p})}{\hbar} \quad (21)$$

Thus, de Broglie anticipated this my work and ontological wavefunction Ψ in his article [16]:

"Above, was only introduced the Ψ wave, containing a very small singular region, generally in motion, which constitutes the particle... It defines the particle's internal structure. We will not insist on this point the study of which at the time being seems premature. It looks quite natural that the propagation in space and time of the truly physical Ψ wave should determine, as was assumed in the guidance theory, the particle's motion, as it is integrated into the wave... However, since the publication of Schrödinger's works in 1926, it became customary to only consider the Ψ wave, of arbitrarily normed amplitude. But this wave cannot be considered as a physical wave..." pp.13-14,

Note that this representation of the particle's matter-density wave-packet is analog to the representation of the perfect isotropic fluid with density $\rho = \Phi_m$ and pressure $\vec{p}$ which is proportional to the matter-density speed $\vec{w}(t, \vec{r})$ (i.e., there is no viscosity, but the pressure is not equal in the 3 spatial directions). Moreover, the integral of $\vec{W}$ over the matter-distribution of the particle gives the particles velocity (the velocity of the particle's barycenter). According to GR, the metric of time-space is not given once and for all, but is a dynamical field interacting with matter, acting and being acted upon. This dynamic nature of the metric is the most important insight of principle of GR.

Let us denote the finitely small volume of the whole wave-packet of massive particle with rest-mass m_0 , in a given instance of time t

, by V_t . Consequently, we obtain that the time-invariant energy in a given inertial reference frame is obtained by integration,

$$1_\Phi \equiv \int \Phi_m(r_4) dV = \int_{r_4 \in V_t} \Phi^2(r_4) dV = m_0 c^2 \quad (22)$$

For any massive elementary particle that propagates in the 3D space with a velocity that changes in the time, because of external forces that influence this particle, the 3D wave-packet distribution $\Phi(r_4)$ changes as well, but it must satisfy the following *auto-conservation* properties:

The wave-packet of an elementary particle do not undergo a spreading, also when it changes its matter density distribution (i.e., its energy-density Φ_m), and tends to its stable stationary distribution in the vacuum far from another elementary particles. That is, the matter has some *internal self-gravitational autocohesive force* analogously to the peace of fluid in the vacuum, so that at any instance of time, the 3D space topology of the matter distribution, and consequently its compressible energy-distribution Φ_m is simply-connected, closed, continuous and differentiable.

The simply connected matter distribution at any fixed instance of time t (the space where its matter/energy distribution in the 3D spacelike hypersurface $\Sigma_t \subset \mathcal{M}$ is greater than zero) means that every closed 3D loop in it, such that for each point $r_4 \in V_t \subset \Sigma_t$ of this loop, $\Phi(r_4) > 0$, can be deformed continuously to a point. A simply-connected space is alternatively characterized by the property if C_1 and C_2 are two open curves connecting two points in this space, then C_1 can be continuously deformed into C_2 . The assumption of the simply-connected space topology for the matter distribution is based on the fact that the autocohesive self-gravitational matter force does not permit the formation of the vacuum bubbles inside the matter/energy distribution³. Thus, there is a finite closed external surface S_t (in Σ_t) of matter-density, such that for each point r_4 inside the volume closed by S_t , $\Phi_m(r_4) \neq 0$, while for each point outside S_t , $\Phi_m(r_4) = 0$. The plausible explanation of the nature of this autocohesive self-gravitational matter force can be based on the elastic nature of the matter-density which can be expanded and compressed as well by generation of an internal potential energy V .

³Analog problems of compressible fluid mechanics, as it was emphasized by many authors, show that vacuum is major difficulty as far as existence of strong solutions for the Navier-Stokes equations is concerned. From the three compressible isentropic Navier-Stokes equations, here we are interested only on the first one, $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{w}) = 0$, where $\rho(t, \vec{r})$ is the gas/fluid density and $w(t, \vec{r})$ the 'velocity field'. In fact, it will be demonstrated that this continuity equation is valid for the matter/energy density $\Phi_m(r_4)$ of a particle.

For the massive particles we consider that the matter is never entirely within Schwarzschild radius, also when strongly compressed by external forces, so that in the classical solutions of Einstein's theory it should be not a black hole (differently from the point-like theory for the massive particles). The equilibrium of the autocohesive forces is obtained during the inertial propagation (a free particle), when this internal compression/extension potential energy is equal to zero.

Remark: This topological assumption is very important, so that at each instance of time the 3D submanifold V_t of the spacelike hypersurface Σ_t of a pseudo-Riemannian 4D time-space manifold $\mathcal{M}$, is a *simply-connected closed 3-manifold* where we can put a Riemannian metric h (which is a 3×3 submatrix of the pseudo-Riemannian manifold metric g , by taking out the first row and first column. That means, than any possible deformation V_t of the particle's energy-density distribution must satisfy this topological property (in what follows we will consider the 'radial expansions' of the energy-density distributions connected with unstationary particle's states and accelerations. In fact, from the Poincare conjecture (demonstrated recently by Grigoriy Perelman, [11-13]):

"Every simply connected closed 3-manifold is homeomorphic to the 3-sphere",

we have the fact that in stationary cases when a massive particle propagates with a constant speed (w.r.t. a given Minkowski frame), the natural topology of the energy-density distribution V_t is a sphere, like the topology of stars in universe. During acceleration this 'geometry' of V_t changes, but with autocohesive forces when the particle again returns in its inertial propagation, its 'geometry' again becomes perfectly a sphere (3D space symmetry). The physical explanation of this 'implosion' process can be, for example, provided by using the Ricci flow, defined by Richard Hamilton [14].

□

In fact, can be demonstrated that the stable stationary energy-density Φ_m of a massive particle during its inertial propagation in the vacuum (locally flat Minkowski time-space) is spherically symmetric in a sphere with a radius r_0 with radial distribution proportional to $\frac{1}{r^2}$, for $r \leq r_0$. In the unstable states (during interaction of a particle with external fields) we have the spatial extension of Φ_m which again reduces to this spherically symmetric stable distribution in a sphere with radius r_0 .

We define the barycenter as the position of the center of the matter of a given particle in a time-instance t in the Minkowski time-space of the quantum laboratory with observes, corresponding to the particle's trajectory as it is measured by an observer,

$$\vec{r}_T(t) \equiv \frac{1}{\mathbf{1}_\Phi} \int_{r_4 \in V_t} \vec{r} \Phi^2(r_4) dV \quad (23)$$

where $\vec{r}$ is the corresponding space-vector (expressed in the observer's Euclidean space) of the 4D point $r_4 \in V_t \subset \Sigma_t$ of the time-space manifold $\mathcal{M}$ in which Φ_m is a scalar field, with the following assumptions:

1. It is also the center of the mass for the massive particles, and the

vector velocity $\vec{v} = \frac{d\vec{r}_T(t)}{dt}$ corresponds to this barycenter

position, so that the trajectory of a particle is just the trajectory of this barycenter and $\vec{v}$ is tangent on it. The transfer of the

energy/mass of a particle is effected by the barycenter. During inertial propagation, each infinitesimal amount of particle's density Φ_m propagates by the same constant speed $\vec{v}$ as particle's barycenter.⁴

2. The limit of the light velocity is applied to the barycenter (energy transfer); there is no velocity limitation for the density distribution, if its effect to the barycenter is coherent with the previous assumptions.

⁴During the accelerations this is not so, and in that case we introduce the energy-flow velocity vector $\vec{w}(r_4)$ in Definition 2.

The real scalar field Φ of the particle's wave-packet propagates in the ordinary 3D space with a velocity $\vec{v} = v_x e_1 + v_y e_2 + v_z e_3$ (which is the *group speed* of particle's wave-packet), where e_i are the unit orthonormal vectors of the Minkowski time-space basis. In the stationary case, when it propagates in the vacuum with a constant velocity, it has a constant distribution, that propagates as a wave-packet.

This assumption is in accordance both with the classical mechanics and special relativity: The mass/energy is transferred by the particle barycenter, so the velocity of a particle, which lies on the barycenter, can not be greater than the velocity of light. But the changing of the energy-density distribution Φ_m of a particle, which lies in the plain perpendicular to the trajectory (i.e., to the vector velocity $\vec{v}$ has no such a limitation and, for example, if it does not produce the significant changes of the barycenter velocity, we may have quasi instantaneous 'explosions' of the density propagation in this perpendicular plane. It is significant in the case of the 'tunneling effects' of the particles, when they find a material barriers in direction of their propagation: in such cases their distribution changes with enormous velocity (without speed of light limit) in the perpendicular directions to the particle trajectory, so if in the barrier we have the slits also in significant distances, the matter distribution will reach quasi instantly these slits and pass through *all of them* contemporarily into other side of the barrier, where this particle will recompose its compact distribution again (thanks to the autocohesive self-gravitational matter properties). From the fact that such a radial 'explosion' do not change significantly the barycenter, it will appear that the continuous trajectory of the particle passes directly through the barrier as 'tunneling effect'.

The next assumption considers the basic energy-density properties during accelerations: Internal Dynamics Assumption: The matter-density has a kind of the following properties of the fluids (non relativistic case):

1. The matter-density distribution, for a real constant $\angle$ of relationship of the matter-density with its rest-mass density,

$$\angle \Phi_m(r_4) = \angle \Psi \bar{\Psi} = \angle \Phi^2(r_4) \text{ at a point } r_4 \in \mathcal{M}$$

with compressible fluid dynamics; The extension or the matter-density volume (generated by the locally-strong external to this particle 4-D time-space curvature), w.r.t. the stationary stable particle state when a free particle propagates in the vacuum with a constant speed, generates an internal self-gravitational potential energy V by changing particle's kinetic energy (particle's accelerations).

2. The compressible matter/energy-flow velocity vector $\vec{w}(t, \vec{r})$, in the Minkowski time-space M^4 coordinate system, of the infinitesimal amount of the matter at point $r_4 = (t, \vec{r})$, introduced as a 3-dimensional 'velocity field' at a given point r_4 , must satisfy the following conservation law for the matter:⁵

⁵In the case of an inertial propagation with a constant speed $\vec{v}$ and hence $\vec{w} = \vec{v}$, from this conservation law we obtain $\frac{\partial \Phi_m}{\partial t} = \vec{v} \nabla \Phi_m$. Thus, by substitution $\Phi_m(r_4)$, we obtain $\frac{\partial \Phi}{\partial t} = \vec{v} \nabla \Phi$ which is exactly the equation in (27).

$$\frac{\partial \Phi_m}{\partial t} + \nabla \cdot (\Phi_m \vec{w}(t, \vec{r})) = 0 \quad (24)$$

In fact, from time-invariance of particle's rest-mass, we have that $0 = \frac{d}{dt} \mathbf{1}_\Phi = \frac{d}{dt} \int \Phi_m dV = \int \frac{\partial \Phi_m}{\partial t} dV$. We can substitute $\frac{\partial \Phi_m}{\partial t}$ by $-\nabla \cdot (\Phi_m \vec{w}(t, \vec{r}))$, because from Gauss theorem, in any fixed time-instant t , from the finiteness assumption there is a finite sphere with the surface S which contains particle's volume V_t , so that $\int \nabla \cdot (\Phi_m \vec{w}(t, \vec{r})) dV = \int_{V_t} \nabla \cdot (\Phi_m \vec{w}(t, \vec{r})) dV = \oint_S \Phi_m \vec{w}(t, \vec{r}) \vec{dS} = 0$. Viceversa, the time-invariance $\frac{d}{dt} \mathbf{1}_\Phi$ is directly derived in this way from the conservation law (24).

Moreover, the following property is satisfied [21, 24] for the particle's velocity:

$$\vec{v}(t) \equiv \frac{d}{dt} \vec{r}_T(t) = \frac{1}{\mathbf{1}_\Phi} \int_{V_t} \Phi_m(t, \vec{r}) \vec{w}(t, \vec{r}) dV \quad (25)$$

and variation of the matter-density speed $\vec{u}(t, \vec{r})$ has to satisfy the following property:

$$\int_{V_t} \Phi_m(t, \vec{r}) \vec{u}(t, \vec{r}) dV = 0 \quad (26)$$

Remark: Let V be an internal potential energy caused by external locally-strong 4-D time-space curvature (contrasting the autocohesive self-gravitational forces) which changes the massive-particle's matter-density from its natural equilibrium distribution. In the case of the stationary particle's state, when it propagates in the vacuum with a constant speed, we have that $V = 0$ (matter-density natural equilibrium) because the external time-space curvature is zero (when this particle propagates in locally flat Minkowski time-space around it). Consequently, the expansion of a massive particle makes lower value of the kinetic energy, and hence determines a deceleration of this particle. The reduction of an expanded particle toward its equilibrium stable state produces the inverse process: this potential

energy is transformed into kinetic energy, and hence determines the acceleration and augmentation of the total energy of a particle.

□

Based on the conservation law (24), we obtain the following definite partial differential equations (PDE) [21, 24]:

Proposition 1 Let us consider the general, also non-stable, particle's state with the accelerations on its trajectory and the velocity $\vec{v}(t)$ of its barycenter. Then, we obtain the following PDE:

$$\begin{aligned} \frac{\partial \Phi_m}{\partial t} &= -\nabla \cdot (\Phi_m \vec{w}) \\ \frac{\partial \Phi}{\partial t} &= \vec{w} \nabla \Phi - \Phi \frac{\nabla \cdot \vec{w}}{2} \\ \frac{\partial \Psi}{\partial t} &= -i\omega_p \Psi + \vec{w} \nabla \Psi - \Psi \frac{\nabla \cdot \vec{w}}{2} \end{aligned} \quad (27)$$

where $\omega_p = \frac{\partial \varphi_T}{\partial t}$ and

$$\vec{w}(t, \vec{r}) = \vec{v}(t) + \vec{u}(t, \vec{r}) \quad (28)$$

is the (total) velocity of each piece of matter at a position $(t, \vec{r})$, while $\vec{u}(t, \vec{r})$ is its speed relative to particle's barycenter $\vec{r}_T(t)$.

Based on obtained PDEs in (27), we are able to introduce [23, 24] the following TSPF selfadjoint (Hermitian) operator:

Theorem 1 Given a trajectory $\vec{r}_T(t)$ and a vector field $\vec{w}(t, \vec{r})$ for $t \geq 0$, let us define the linear 'time-space-perturbation-flux' (TSPF) operator:

$$\widehat{\mathbb{M}} \equiv i\hbar(\vec{v}(t) + \vec{u}(t, \vec{r})) \nabla - i\hbar \frac{\nabla \cdot \vec{u}(t, \vec{r})}{2} \quad (29)$$

where $\vec{v}(t) = \frac{d\vec{r}_T}{dt}$ and $\vec{u}(t, \vec{r}) = \vec{w}(t, \vec{r}) - \vec{v}(t)$.

This operator is selfadjoint (Hermitian) with the infinite set of real continuous eigenvalues $-\hbar \vec{k} \vec{v}(t)$ for each wavenumber (index) $\vec{k}$ and corresponding complex eigenfunctions

$$\Phi_{\vec{k}}(t, \vec{r}) = \frac{c_0}{(\sqrt{2\pi})^3} e^{h_0(t, \vec{r})} e^{ih_1(t, \vec{r}) - i\vec{k} \vec{r}} \quad (30)$$

where the real dimensionless functions h_0 and h_1 are determined by the equations:

$$(\vec{v} + \vec{u}) \nabla h_1(t, \vec{k}, \vec{r}) = -\vec{k} \vec{u}, \quad (\vec{v} + \vec{u}) \nabla h_0(t, \vec{r}) = \frac{\nabla \cdot \vec{u}}{2} \quad (31)$$

An important subset of solutions, when we can separate the time and position in the eigenfunctions with $\Phi_{\vec{k}}(t, \vec{r}) = T_{\vec{k}}(t)R_{\vec{k}}(\vec{r})$, can be obtained in the cases when there exist two real dimensionless functions $h_0(\vec{r})$ and $h_1(\vec{k}, \vec{r})$ such that the following equations are satisfied

$$(\vec{v} + \vec{u})\nabla h_1(\vec{k}, \vec{r}) = -\vec{k}u, \quad (\vec{v} + \vec{u})\nabla h_0(\vec{r}) = \frac{\nabla \cdot \vec{u}}{2} \quad (32)$$

and hence we obtain the following set of the eigenfunctions

$$\Phi_{\vec{k}}(t, \vec{r}) = \frac{c_0}{(\sqrt{2\pi})^3} e^{h_0(\vec{r})} e^{ih_1(\vec{k}, \vec{r})} e^{-i\vec{k}(\vec{r} - \vec{r}_T(t))} \quad (33)$$

where $c_0 = (\sqrt{2\pi})^3 (\int e^{2h_0(\vec{r})} dV)^{-1/2}$. In the simplest case, when $\vec{v}$ is constant and $\vec{u} = 0$, we have that h_0 and h_1 are equal to zero and $c_0 = 1$ in equation (33).

Then, the time evolution of massive particle's shape $\Phi(t, \vec{r})$ which propagates on the trajectory $\vec{r}_T(t)$, heaving the internal energy-density speed (w.r.t. it barycenter) $\vec{u}(t, \vec{r})$ and a given initial shape $\Phi(0, \vec{r})$ at initial position $\vec{r}_0 \equiv \vec{r}_T(0) = \int \vec{r} \Phi(0, \vec{r}) dV$, is defined by the following differential equation for any $t > 0$:

$$i\hbar \frac{\partial}{\partial t} \Phi(t, \vec{r}) = \hat{\mathbb{M}} \Phi(t, \vec{r}) \quad (34)$$

where PDE (??) is equal to the second PDE in (27). It is shown that the average value (expectation value) of the TSPF operator is

$$\hat{\mathbb{M}}_{av} = \langle \Phi | \hat{\mathbb{M}} | \Phi \rangle = \int \Phi(\hat{\mathbb{M}}\Phi) dV = 0 \quad (35)$$

It is shown [23, 24] that the TSPF operator $\hat{\mathbb{M}}$ has the Hilbert space with orthonormal basis:

$$B_{\mathcal{H}} \equiv \{ \Phi_x(t, \vec{r}) = \frac{c_0}{(\sqrt{2\pi})^3} e^{h_0(\vec{r})} e^{ih_1(x\vec{v}(0), \vec{r})} e^{-i\vec{x}\vec{v}(0)(\vec{r} - \vec{r}_T(t))} | x \in \mathbb{R} \} \quad (36)$$

with the constant $c_0 = (\sqrt{2\pi})^3 (\int e^{2h_0(\vec{r})} dV)^{-1/2}$ for an accelerated particle, while for a free particle $c_0 = 1$ and the functions h_0 and h_1 are the constants equal to zero.

Thus, based on the infinite o.n.b. (??), any particle's body shape Φ can be expressed by the expression

$$\Phi(t, \vec{r}) = \int_{x \in \mathbb{R}} b_x \frac{c_0}{(\sqrt{2\pi})^3} e^{h_0(\vec{r})} e^{ih_1(x\vec{v}(0), \vec{r})} e^{-i\vec{x}\vec{v}(0)(\vec{r} - \vec{r}_T(t))} dx \quad (37)$$

with the coefficients $b_x \in \mathbb{C}$.

We can show [23, 24] that obtained particle shape Φ is really a real function: the superpositions $\Phi(t, \vec{r})$ of the o.n.b. $B_{\mathcal{H}}$, defined in (??), with the coefficients $b_x \in \mathbb{C}$, for each $x \in \mathbb{R}$, can be expressed by the following integral of *only real functions*, by using the polar expression of the complex coefficients $b_x = |b_x| e^{i\beta_x}$,

$$\Phi(t, \vec{r}) = \frac{2c_0}{(\sqrt{2\pi})^3} e^{h_0(\vec{r})} \int_{x>0} |b_x| \cos(h_1(x\vec{v}(0), \vec{r}) - x\vec{v}(0)(\vec{r} - \vec{r}_T(t)) + \beta_x) dx \quad (38)$$

We have seen that in the stationary cases, during inertial propagation in the vacuum, is easy to represent the trajectories of the matter (the barycenter) of the particles. The question about what is the trajectory of a particle in the unstationary situations (necessarily during accelerations), when the shape of the wave-packet Φ changes (also drastically in quasi infinitesimal amounts of time [20,25]), has to be examined in details in what follows, and this requires the formal definition of the trajectory position linear operators for the particles, and examination of them in such dynamic unstationary conditions:

Proposition 2 Let $\hat{r}_T = \hat{q}_1 e_1 + \hat{q}_2 e_2 + \hat{q}_3 e_3 \equiv \frac{\vec{r}}{\mathbf{1}_{\Phi}}$ denote the linear position operators and let $|\Phi\rangle$ be a vector ket of the wave-packet of a particle. where each $\hat{q}_i$ is an operation of multiplication by $\frac{q_i}{\mathbf{1}_{\Phi}}$ for each Cartesian coordinate $q_i \in \{x, y, z\}$. So, the barycenter of this particle is given by the average values of these position operators:

$$(x_T(t), y_T(t), z_T(t)) = (\langle \Phi | \hat{q}_1 | \Phi \rangle, \langle \Phi | \hat{q}_2 | \Phi \rangle, \langle \Phi | \hat{q}_3 | \Phi \rangle) \in \mathbb{R}^3$$

, that is,

$$\vec{r}_T = \hat{r}_{Tav} \equiv \langle \Phi | \hat{r}_T | \Phi \rangle \quad (39)$$

This new position operator is a generalization of the Schrödinger's position operator and hence satisfies the commutativity relationship

$$\text{with the Schrödinger's momentum operators } \hat{p}_i = -i\hbar \frac{\partial}{\partial q_i},$$

$$[\hat{p}_i, \hat{q}_i]_{av} = (\hat{p}_i \hat{q}_i - \hat{q}_i \hat{p}_i)_{av} = i\hbar, \quad i = 1, 2, 3 \quad (40)$$

Hence, the trajectory of the barycenter of a particle is an 'expected value' or average value of the position operators computed from the trajectories of all point-like matter's pieces of the wave-packet by

$$\vec{r}_T = \langle \Phi | \hat{q}_1 | \Phi \rangle e_1 + \langle \Phi | \hat{q}_2 | \Phi \rangle e_2 + \langle \Phi | \hat{q}_3 | \Phi \rangle e_3.$$

The computation of such expected values of the trajectory and particle's velocity of a given particle is done in the same way as the expected (average) values of the energies computed by Hamiltonian operators of the Schrödinger's differential equation. The difference is only that instead of the 'energy-type' of the Schrödinger's hamiltonian

operator $\hat{H}$ and its corresponding eigenvalues (representing the sum of the kinetic and potential energy levels), here we are using a new 'trajectory-type' position linear TSPF operator $\hat{\mathbb{M}}$, given by equation (29), with a continuous infinite spectra of the eigenvalues.

More about quantum operators in this IQM theory, as momentum and energy operators, is provided in [23, 24].

Strong Completion of Quantum Mechanics

In last publications [23, 24] we have shown that the QM is composed by two mutually complementary point of views: the ontological IQM theory of real individual particles represented by their rest-mass energy densities and ontological complex function Ψ , and from the SQM statistical theory based on an epistemic point of view based on the *quantum ensemble* where a quantum state is defined as an ensemble of identically prepared systems, with a density matrix representing a mixed state (where Ψ is Schrödinger wavefunction).

However, these two complementary representations of QM theory are not strongly unified in a single complete QM theory in which the statistical Schrödinger equation is derived from the ontological IQM theory of individual particles. So, in what follows we will present this strong unification in order to obtain a *strong completion* of Quantum Mechanics.

In fact, it has been demonstrated [23] that in the IQM theory each single prepared setup, as a part of an ontological ensemble, corresponds to *an individual* particle which propagates with its unique trajectory $r_T(t)$ during the measurement. The result of the measurement of all prepared setups in a given ensemble will give the statistical probabilistic average result of measurements. So, we obtain a clear physical explanation of the relationship between the statistical SQM with quantum ensemble and deterministic IQM theory of the quantum mechanics with ontological ensemble. This question has been anticipated by Luis de Broglie in 1987:

"It is nevertheless unquestionable that use of the Ψ wave and its generalization did lead to accurate prediction and fruitful theories. This is an indisputable fact. The situation is clarified by introducing together with the statistical Ψ wave, the Ψ wave, which being an objective physical reality, may give rise to phenomena the statistical aspect of which is given by the Ψ wave."

It becomes important to establish the relationship between the Ψ and Ψ waves... These arguments present great similarity with those used by Einstein and his co-workers to justify in General Relativity the statement that a material particle moves along a space-time geodesic. " pp.13-14,

Note that the continuity equation (24) of the rest-mass energy density $\Phi_m(t, \vec{r})$ of an individual massive particle has the same mathematical form of the Liouville's equation [6] for the divergenceless flows in (10), which is just a version of the Noether continuity equation for the conserved current where the *phase space probabilistic density* function f , such that at each instance of time t holds $\int_M f(t, x) dx = 1$. So if we work with normalized energy

density $\frac{1}{m_0 c^2} \Phi_m(t, \vec{r})$, the only difference between them is that in our case we are working in real 4D Minkowski timespace with the points $(t, \vec{r})$ while the probability density f is in phase space and can be defined as the relative number density of the phase points associated with an infinite statistical ensemble.

This is just a good indication that, also in our case, we can introduce the classical ensemble for observed elementary particles during the set of repetitive measurements. It will be provided in what follows by equation (74). Such classical ensemble of repetitive measurements on the set of individual particles need the following definition of normalized multi-particle system.

Normalized multi-particle system for quantum ensemble:

The EI (an ensemble of similarly prepared measurement systems) of SQM claims that it is *minimalist*, making the fewest assumptions about the meaning of the Schrödinger wavefunction Ψ , and proposes to take to the fullest extent the statistical interpretation of Max Born. A fundamental principle of SQM is that only probabilistic statements can be made, whether for individual systems/particles, a simultaneous group of systems/particles, or a collection (ensemble) of systems/particles.

The ensemble is not a single preparation and observation of one simultaneous set of particles. The members of a quantum ensemble are said to be in the same state Ψ (Schrödinger complex wavefunction), and this state Ψ (a *state vector* $|\Psi\rangle$ in Hilbert space is not unique, because any vector $e^{i\alpha} |\Psi\rangle$, for a real α , is equivalent to $|\Psi\rangle$) can be mathematically denoted by a statistical *pure state operator* $\hat{\rho} = |\Psi\rangle\langle\Psi| = \int \Psi(_) \overline{\Psi(_)} dV$ which is a function from Hilbert space to Hilbert space and may be written as a density matrix. Quantum Ensemble:

Let, for simplicity, the Hilbert orthonormal base $\mathcal{B}_H$ of Hamiltonian operator $\hat{H}$ be a finite set with $n \geq 2$ of its eigenfunctions e_i , such that $\hat{H}e_i = H_i e_i$ for $1 \leq i \leq n$.

Then, for this quantum ensemble the mixed state operator, used for the statistical mixture of the states (for example, it is either $|e_j\rangle$ or $|e_k\rangle$, but we do not know which one), with its spectral representation

$$\hat{\rho} = \sum_{i=1}^n P_i |e_i\rangle\langle e_i| \quad (41)$$

is the Hermitian (self-adjoint) one with positive eigenvalues P_i such that

$$\sum_{i=1}^n P_i = 1 \quad (42)$$

that is P_i are the probabilities, $0 < P_i \leq 1$. So, differently from the superposition of pure states $\sum_{i=1}^n c_i e_i$, with $c_i \in \mathbb{C}$, where the probability is quantum intrinsic, for quantum ensemble the probability is classically *statistic*. The problem in SQM is that the eigenfunctions $e_i(t, \vec{r})$ of the Hamiltonian operator are, at any fixed time t , present practically in the whole 3-D space, so that each pair of them are mutually superposed with mutual interference generating an "intrinsic randomness". So we can not use the superposition of pure states $\sum_{i=1}^n c_i e_i$ as representation of ensembles where the reason of randomness of measurement outcomes is just classical statistical during the measurements in which the initial states of each individual measurement are unknown.

This problem does not exist for the particle's ontological wavefunctions $\Psi_i(t, \vec{r})$ which are spatially always delimited in small 3-D volumes in which is present particle's rest-mass energy-density, and if we have no fusion (with generation of new particles) of two elementary particles, like during ensemble of measurement: we have the *separability property* of individual particles and their ontological wavefunctions.

So in SQM, the quantum ensemble is represented by this mixed state operator in (41). The expectation value of any operator $\hat{A}$ is then defined as $(\hat{A})_{av} = Tr(\hat{\rho}\hat{A})$ which for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, gives the usual result $(\hat{A})_{av} = Tr(|\psi\rangle\langle\psi|\hat{A}) = \langle\psi|\hat{A}|\psi\rangle$. For the mixed states, it gives

$$(\hat{A})_{av} = Tr(\sum_{i=1}^n P_i |e_i\rangle\langle e_i| \hat{A}) = \sum_{i=1}^n P_i \langle e_i | \hat{A} | e_i \rangle \quad (43)$$

Thus, for the Hamiltonian operator $\hat{H}$ in Definition 3, its expectation value of the quantum ensemble is

$$(\hat{H})_{av} = Tr(\sum_{i=1}^n P_i |e_i\rangle\langle e_i| \hat{H}) = \sum_{i=1}^n P_i \langle e_i | \hat{H} | e_i \rangle = \sum_{i=1}^n P_i H_i \quad (44)$$

We need to define the ontological counterpart to the quantum ensemble with the set of particle's Ψ_i , for $1 \leq i \leq N$, and $N \gg 1$ used in such repetitive measurements, so that such physical system is just a *particular multi-particle system*, in which for a given interval Δt , with $t = 0$ be the beginning of this time-interval, of this total measurements process, for each time instance $t \in \Delta t$ only one of these particle's is under laboratory measurement.

Let this set of particles observed in a enough high number $N \gg 1$ of repetitive measurements, so that in these repetitive measurements for each eigenvalue H_i of the Hamiltonian operator $\hat{H}$ used in Schrödinger's equation there is at least one observed particle with this total energy, and denote by

$$\mathcal{B}_N(\Delta t) = \{\Psi(t, \vec{r}) | t \in \Delta t\} \quad (45)$$

the set of all $N \gg 1$ observed particles during the time Δt of the whole process of repetitive measurements.

In order to formalize this mutual-independence, that is, that fact that their energy-density volumes $V_i(t)$ are not mutually superposed, that is, for $1 \leq i, k \leq n$, we have the *separability property*

$$V_i(t) \cap V_k(t) = \emptyset \quad \text{if } i \neq k, \text{ for all } t \in \Delta t \quad (46)$$

That is, from the orthogonal property, for $\Psi_i, \Psi_k \in \mathcal{B}_n$, at each 4-D time-space point $(t, \vec{r})$ from (46) we have that for each $t \in \Delta t$,

$$\Psi_i(t, \vec{r}) \Psi_k(t, \vec{r}) = 0 \quad \text{if } i \neq k \quad (47)$$

However, the multi/particle vector base $\mathcal{B}_n(\Delta t)$ is not normalized because it holds that, for each $t \in \Delta t$,

$$\langle \Psi_i | \Psi_k \rangle = \int \bar{\Psi}_i(t, \vec{r}) \Psi_k(t, \vec{r}) dV = m_i c^2 \delta_{ik}, \quad (48)$$

where $\delta_{ik} = 1$ if $i = k$; 0 otherwise, is Kronecker function.

Now, if we consider an individual elementary particle (in a multi-particle system defined by orthogonal vector space $\mathcal{B}_n(\Delta t)$ in (50)) represented by its ontological wavefunction $\Psi_i \in \mathcal{B}_n(\Delta t)$, but now represented by its *normalized ontological wavefunction* $\psi_i^o(t, \vec{r})$ in (53) the equation (48) reduces to the orthonormal form

$$\langle \psi_j^o | \psi_k^o \rangle = \left\langle \frac{\Psi_j}{\sqrt{m_j c^2}} \middle| \frac{\Psi_k}{\sqrt{m_k c^2}} \right\rangle = \delta_{jk} \quad (49)$$

so that $\langle \psi_i^o | \psi_i^o \rangle = 1$.

However, the orthogonal vector base $\mathcal{B}_N(\Delta t)$ is *degenerate* because we have more than one eigenvector with the same Hamiltonian, so it is convenient to reduce it into a subset such that there are no more than one eigenvector for each Hamiltonian H_i used in Definition 3 of quantum ensemble for $1 \leq i \leq n$. So, we can define any subset of n representative observed particles,

$$\mathcal{B}_n(\Delta t) = \{\Psi_i(t, \vec{r}) | 1 \leq i \leq n, t \in \Delta t\} \subset \mathcal{B}_N(\Delta t) \quad (50)$$

such that the total energy of particle Ψ_i is equal to the Hamiltonian H_i used in Definition 3 of quantum ensemble, in the way that $\mathcal{B}_n(\Delta t)$ covers all spectra of eigenvalues of Hamiltonian operator $\hat{H}$ used to define the quantum ensemble.

This set $\mathcal{B}_n(\Delta t)$ is multi-particle system, which is a collection

of $n \geq 2$ individual particles such that there is no any kind of fusion between them, that is, each of them for a given instance of time have no superposition of its energy-density with other individual particles of this multi-particle system.

Thus, we are able to define the multi-particle system with the following orthonormal basis: By considering non normalized vector base $\mathcal{B}_n(\Delta t)$ in (50), with $n \geq 2$ elements, we define the following orthonormal vector base:

$$\mathcal{B}_p(\Delta t) = \{\psi_i^o = \frac{\Psi_i}{\sqrt{m_i c^2}} | \Psi_i \in \mathcal{B}_n(\Delta t)\} \quad (51)$$

so that we have the following bijection with the Hilbert vector base $\mathcal{B}_H$ in Definition 3 of quantum ensemble,

$$\sigma: \mathcal{B}_p(\Delta t) \simeq \mathcal{B}_H \quad (52)$$

such that $\sigma(\psi_i^o) = e_i$, for $1 \leq i \leq n$. which will be used in next for definition of *ontological ensemble*, as dual representation of quantum ensemble in Definition 3 (usually all observed particles in such repetitive measurements have the same rest-mass).

Derivation of Schrödinger Equation from the Deterministic IQM Theory

Consequently, the first step in derivation of statistical QM Schrödinger equation (with its complex normal wavefunction Ψ) from the deterministic IQM theory with hidden variables of individual particles, is to define a complex normalized ontological classical wavefunction ψ^o from the complex wave packet (17) of an elementary particle with rest-mass m_0 , $\Psi = \Phi(t, \vec{r}) e^{-i\varphi_T}$, as provided in by vector base in (51), $\mathcal{B}_p(\Delta t)$,

$$\psi^o(t, \vec{r}) \in \mathcal{B}_p(\Delta t) \quad (53)$$

which now is a normalized space-perturbation (abstract particle's mathematical density forms with no physical dimension) of an individual particle.

The Hilbert space and scalar (inner) product for such classical wavefunction ψ^o we can provide from its *Hermitian* (self-adjoint) normalized operator $\hat{\mathbb{M}} = [\hat{\mathbb{F}}]$ introduced in [23] (in Section 4.6) and in [24] (in Section 7.4.2) respectively, from the third PDE in (27). It is demonstrated that this fundamental operator $\hat{\mathbb{F}}$ is Hermitian (self-adjoint) obtained from hidden particle's variables $\vec{w}(t, \vec{r})$ (internal particle's energy-density flux) which, for the ontological wavefunction of the particle, has a simple form

$$\hat{\mathbb{F}} \equiv \hbar \omega_p + \hat{\mathbb{M}} \quad (54)$$

where the for each instance of time $\omega_p = \frac{\partial}{\partial t} \varphi_T$ is a real number and

$$\hat{\mathbb{M}} = i\hbar \vec{w}(t, \vec{r}) \nabla - i\hbar \frac{\nabla \cdot \vec{w}(t, \vec{r})}{2} \quad (55)$$

is the TSPF Hermitian operator in (29), so that from the third PDE in (27) we obtain the fundamental ontological wavefunction equation

$$i\hbar \frac{\partial}{\partial t} \psi^o = \hat{\mathbb{F}} \psi^o \quad (56)$$

The orthonormal vector basis for the Hilbert space $\mathcal{H}^o$ of this time-dependent Hermitian linear operators $\hat{\mathbb{M}}$ and $\hat{\mathbb{F}}$ is, from [23] (in Section 4.6) and in [24] (in Section 7.4.2) respectively,

$$\mathcal{B}_{\mathcal{H}^+} \equiv \{\Psi_x = \frac{c_0}{(\sqrt{2\pi})^3} e^{h_0(\vec{r})} e^{ih_1(x\vec{v}(0), \vec{r})} e^{-ix\vec{v}(0)(\vec{r}-\vec{r}_T(t))} e^{-i\varphi_T} | x \in \mathbb{R}\} \quad (57)$$

that is, it is equal to that of the TSPF operator given in (??) multiplied by particle's phase component $e^{-i\varphi_T}$ in (53), with the constant $c_0 = (\sqrt{2\pi})^3 (\int e^{2h_0(\vec{r})} dV)^{-1/2}$ for an accelerated particle, while for a free particle $c_0 = 1$ and the functions h_0 and h_1 are the constants equal to zero.

So, based on the vector base $\mathcal{B}_{\mathcal{H}^+}$ above of this Hilbert space $L^2(\mathbb{R}^3)$, the ontological wavefunction ψ^o (whose hidden flux variable $\vec{w}(t, \vec{r})$ is used to define the Hermitian operator $\hat{\mathbb{F}}$ of this individual particle) can be expressed as

$$\psi^o = \int_{b_x} b_x \Psi_x dx \quad (58)$$

with $x \in \mathbb{R}$ and $b_x \in \mathbb{C}$. Note that from the fact that ψ^o represent the normalized particles energy-density, the $|b_x|^2$ of the complex constants in this superposition of orthonormal vectors *can not be interpreted as probabilities*!

However, in such a Hilbert space $\mathcal{H}^o$ of an individual particle with rest-mass m_0 , for its scalar (inner) product we obtain the normalization

$$\langle \psi^o | \psi^o \rangle = \int \overline{\psi^o} \psi^o dV = 1 \quad (59)$$

analog to that for the Schrödinger wavefunctions Ψ in (5). So, given a Hermitian IQM operator $\hat{A}$, the average value of this operator is defined by:

$$\hat{A}_{av} = \langle \psi^0 | \hat{A} | \psi^0 \rangle = \int \overline{\psi^0} \hat{A} \psi^0 dV \quad (60)$$

So, we also need to show that deterministic evolution in time of the ontological wavefunction is equal to that in (7) for the Schrödinger wavefunctions Ψ ; Deterministic evolution in time of the ontological wavefunction for any fixed 3D point $\vec{r}$ is equal to

$$\psi^o(t) = U(t)\psi^o(0) \quad (61)$$

for some $U(t)$ which is unitary for a free particle. **Proof:** In order to show this let us consider the (total) time-evolution of particle's shape Φ , with the particle's velocity $\frac{d\vec{r}_T}{dt}(t) = \vec{v}(t)$, and with the velocity of each infinitesimal amount of particle's rest-mass energy density (28) at a point $\vec{r} = q_1 e_1 + q_2 e_2 + q_3 e_3$ inside the particle's

body, defined by $\frac{d\vec{r}}{dt} = \vec{w}(t, \vec{r}) = \vec{v}(t) + \vec{u}(t, \vec{r})$.

So, we obtain for the total derivation in time that

$$\begin{aligned} \frac{d}{dt} \Phi(t, \vec{r}) &= \frac{\partial}{\partial t} \Phi + \frac{\partial \Phi}{\partial \vec{r}} \frac{d\vec{r}}{dt} = \frac{\partial}{\partial t} \Phi + \frac{d\vec{r}}{dt} \frac{\partial \Phi}{\partial \vec{r}} \\ &= \frac{\partial}{\partial t} \Phi + \vec{w}(t, \vec{r}) \frac{\partial \Phi}{\partial \vec{r}} \\ &= \frac{\partial}{\partial t} \Phi - \vec{w}(t, \vec{r}) \nabla \Phi \\ &= -\Phi \frac{\nabla \cdot \vec{w}}{2} \text{ from the second PDE in (27).} \end{aligned}$$

So, for the total derivation of ontological complex wave-packet

$\Psi(t, \vec{r})$ we obtain:

$$\begin{aligned} \frac{d\Psi}{dt} &= \frac{d}{dt} \Phi e^{-i\varphi_T} = e^{-i\varphi_T} \frac{d\Phi}{dt} + \Phi e^{-i\varphi_T} \frac{d(-i\varphi_T)}{dt} \\ &= e^{-i\varphi_T} \frac{d\Phi}{dt} - i\Psi \frac{d\varphi_T}{dt} \\ &= e^{-i\varphi_T} \left(-\Phi \frac{\nabla \cdot \vec{w}}{2} \right) - i\Psi \frac{d\varphi_T}{dt} \text{ from above} \\ &= -\left(i \frac{d\varphi_T}{dt} + \frac{\nabla \cdot \vec{w}}{2} \right) \Psi \end{aligned}$$

and hence

$$\frac{1}{\Psi} \frac{d\Psi}{dt} = \frac{d}{dt} \ln \Psi = -i \frac{d\varphi_T}{dt} - \frac{\nabla \cdot \vec{w}}{2}$$

where for $t=0$ the phase, denoted by $\varphi_T|_0$ is zero, with

$e^{-i\varphi_T|_0} = 1$, so that by integration in time, for each $t \geq 0$, we obtain

$$\int_0^t \frac{d}{dt} \ln \Psi d\tau = \ln \Psi|_0^t = -i(\varphi_T - \varphi_T|_0) - \int_0^t \frac{\nabla \cdot \vec{w}}{2} d\tau = -i\varphi_T - \int_0^t \frac{\nabla \cdot \vec{w}}{2} d\tau$$

that is, we obtained the deterministic time-evolution

$$\Psi(t, \vec{r}) = \Psi(0, \vec{r}) e^{-i\varphi_T - \int_0^t \frac{\nabla \cdot \vec{w}}{2} d\tau}, \text{ rewritten in compact form}$$

$$\Psi(t, \vec{r}) = U(t, \vec{r}) \Psi(0, \vec{r}) \quad (62)$$

for $U(t, \vec{r}) = e^{-i\varphi_T - \int_0^t \frac{\nabla \cdot \vec{w}}{2} d\tau}$ which is unitary for a free particle

when $\vec{w}$ is constant speed so that $\nabla \cdot \vec{w} = 0$, so that $U(t) = e^{-i\varphi_T}$.

So, by dividing both sides of this equation by $\sqrt{m_0 c^2}$, for each

prefixed point $\vec{r}$, we obtain the time-evolution equation (61) for ontological (classical) wavefunction ψ^o of an individual particle with rest-mass m_0 .

□

However, remains the fundamental physical difference of ontological wavefunction ψ^o and Schrödinger's wavefunction Ψ : the evolution $U(t, \vec{r})$ is unitary only for non accelerated particles and the continuity equations are very different. In fact, if we use the continuity equation of probabilistic density flow from (8), and apply

it to the ontological wavefunction $\psi^o = \sqrt{\rho(t, \vec{r})} e^{-i\varphi_T(t, \vec{r})}$,

where $\rho = \frac{\Phi_m(t, \vec{r})}{m_0 c^2}$, we obtain

$$\vec{j} \equiv \frac{\hbar}{i2m_0} (\overline{\psi^o} \nabla \psi^o - (\nabla \overline{\psi^o}) \psi^o) = \rho \left(-\frac{\hbar}{m_0} \nabla \varphi_T(t, \vec{r}_T(t)) \right) = 0$$

from the fact that $\nabla \varphi_T(t, \vec{r}_T(t))$ because φ_T is not plain-wave phase (with free position variable $\vec{r}$) but is defined only on particle's trajectory $\vec{r}_T(t)$, and hence with continuity equation

$$0 = \frac{\partial}{\partial t} \rho + \nabla \cdot \vec{j} = \frac{\partial}{\partial t} \rho$$

that is, $\frac{\partial}{\partial t} \Phi_m(t, \vec{r}) = 0$, which holds only for a free particle in particular frame in which its speed is zero! Thus, we see how it is important that ontological wavefunction is always local: normalized ontological wavefunctions ψ^o have very different topology from the Schrödinger's wavefunctions Ψ .

Remark: Notice that the ontological classical wave function $\psi^o(t, \vec{r})$ is analog to the complex "wave function" $\psi^c(t, x)$ in phase-space (introduced by Koopman and von Neumann (KvN)[7, 8]) given in (12), where the phase-space probabilistic density function

$f(t, x)$ is replaced by our $\frac{1}{m_0 c^2} \Phi_m(t, \vec{r})$.

Based on the method used by KvN of "quantization" of the statistical Liouville equation to obtain the quantum statistical Schrödinger equation, we will use in an analog process of quantization of our fundamental ontological wavefunction equation (??) in order to obtain from it the statistical Schrödinger equation.

This approach implies that in classical mechanics (IQM theory of individual particles is part of classical mechanics) one can employ operators (like that above in (??)) entirely analogous to those in quantum mechanics. However, from the Hilbert space $\mathcal{H}^o$ of the classical operator $\hat{\mathbb{F}}$ or $\hat{\mathbb{M}}^w = [\hat{\mathbb{F}}]$ for a single individual particle represented by classical ontological wavefunction ψ^o , we *can not* derive the Born rule (see the explanation below equation (58)).

Born rule can emerge only from the repetitive measurement, composed by $N \gg 1$ measurements under practically equal conditions of some observable particle's property. That is, Born rule can emerge only by using such *ontological ensemble*. Thus, in what follows, we will introduce the ontological ensemble and corresponding Hermitian IQM operator $\hat{H}_e$, whose Hilbert space $L^2(\mathbb{R}^3)$ provides the appearance of the probabilities during these repetitive measurements and hence Born rule as well.

□

So, in this ontological description we obtained:

1. 3D nonlinear localized rest-mass energy-density field Φ_m with deterministic internal energy-density flow.
2. Complex ontological wavefunction ψ^o with normalized representation of internal energy-density dynamics.
3. Particle's Hilbert structure of fundamental IQM operator $\hat{\mathbb{F}}$ as a scalar extension of TSPF operator containing the hidden vector variables $\vec{w}(t, \vec{r})$ of internal particle's dynamics.
4. Eigenvector decomposition.
5. Deterministic evolution.
6. Probability might emerge only from ontological ensemble of repeated measurements.

So we need a derivation of quantum probability during measurements of the same kind of particles with the fixed rest-mass m_0 in an *ontological ensemble* which will be provided in next subsection. That is a deeper foundational move. With this, the

quantum mechanics is effective, and in order to show that from this IQM ontological theory of individual particles also the Born rule for the ensemble of measurements is derivable.

Such a framework is rigorous and resembles:

Ensemble $\mapsto$ spectral decomposition $\mapsto$ quadratic probability law.

The strong completion of QM can done by the derivation of the Schrödinger wavefunctions equation from the classical mechanic equation (??) for ontological wavefunctions ψ^o which are complex, normal, and with the same deterministic evolution in time with unitary $U(t)$ as the Schrödinger wavefunctions.

Unique difference from the Schrödinger and ontological equation of their complex wavefunctions are their eigenfunctions which in statistical equation of Schrödinger are derived from the Hamiltonian operator $\hat{H}$ while in classical ontological equation are derived from the IQM fundamental operator $\hat{\mathbb{F}}$.

So, by passing from the classical deterministic equation to quantum statistical equation, we have to use the standard process of 'quantification', as we provided in Section 2 for quantization of classical Liouville wave equation to obtain the quantum statistical Schrödinger equation. So, during this quantization we replace the classical ontological wavefunction ψ^o by the probabilistic Schrödinger wavefunction ψ .

Thus, it remains only to do the quantization of the classical IQM fundamental operator (??) which, by using $\omega_p = \frac{H}{\hbar}$ from (21), becomes

$$\hat{\mathbb{F}} \equiv \hbar \omega_p + \hat{\mathbb{M}} = H + \hat{\mathbb{M}} \quad (63)$$

where H is the Hamiltonian (total energy of this individual particle), and where TSPF operator $\hat{\mathbb{M}}$ is composed by hidden variables (particles flux of energy-density) only. In this quantisation of $\hat{\mathbb{F}}$ we have two steps:

1. From the fact that SQM is rigorously no-hidden-variables theory, we have to forget the TSPF operator component in (??). This is fundamental step to transform a classic equation of an *individual* particle into statistical no-hidden variables Schrödinger equation: particular outcome of each measurement in an ensemble explains why the outcome of a quantum ensemble is probabilistic (statistical result of repeated measurements).
2. After first quantization step, we obtained the transformation $\hat{\mathbb{M}}^w \mapsto H$, so in this second step we only have to quantize the Hamiltonian by replacement of the momentum vector $\vec{p}$ by its quantum operator $-i\hbar\nabla$, as used by Schrödinger in (3). So, we obtain that the final result of quantization of $\hat{\mathbb{F}}$ is

the Hamiltonian operator $\hat{H}$ in (4).

This complete quantization can be represented by transformations from classical to quantum equation:

$$\psi^o \mapsto \psi \quad \text{and} \quad \hat{\mathbb{F}} \mapsto \hat{H} \quad (64)$$

and hence, with this, we derived statistical Schrödinger equation (1) from the classical ontological equation (??).

That this quantization is mathematically correct can be shown by the following corollary:

Corollary 1 Let the quantum ensemble in Definition 3 of identically prepared systems, be represented in SQM by the mixed state operator (41). Then the probabilistic average of the total energy outcomes of this quantum ensemble satisfies, for all $\psi_i^o \in \mathcal{B}_p(\Delta t)$ in Definition 3,

$$(\hat{H})_{av} = \sum_{i=1}^n P_i \langle \psi_i^o | \hat{\mathbb{F}}_i | \psi_i^o \rangle \quad (65)$$

where, from (44), P_i is the probability of the measurement outcome H_i with the "IQM to derived SQM point-to point" relationship, from (52), $e_i = \sigma(\psi_i^o)$,

$$\langle e_i | \hat{H} | e_i \rangle = \langle \psi_i^o | \hat{\mathbb{F}}_i | \psi_i^o \rangle = H_i \quad \text{for each } 1 \leq i \leq n \quad (66)$$

Proof: Note that in EI interpretation (as follows from Section 1) of the quantum ensemble in (41), and from Definition 3, the eigenfunction e_i in o.n.b. of the Hilbert space $\mathcal{H}$ of the Hamiltonian operator H corresponds ontological wavefunction ψ_i^o with its fundamental IQM Hermitian linear operator $\hat{\mathbb{F}}_i$ containing the hidden variables (rest-mass energy density flux $\vec{w}(t, \vec{r})$ inside particle's finite volume).

The well known probabilistic average in (44) of the total energy outcomes of this ensemble is obtained as

$$(\hat{H})_{av} = \sum_{i=1}^n P_i H_i \quad \text{with} \quad \sum_{i=1}^n P_i = 1 \quad (67)$$

so, in order to derive the equation (65) it is enough to show that the total

energy H_i of elementary particle represented by $\psi_i^o \in \mathcal{B}_p(\Delta t)$ is equal to $\langle \psi_i^o | \hat{\mathbb{F}}_i | \psi_i^o \rangle$.

In fact,

$$\begin{aligned} \langle \psi_i^o | \hat{\mathbb{F}}_i | \psi_i^o \rangle &= \int \bar{\psi}_i^o \hat{\mathbb{F}}_i \psi_i^o dV = \int \bar{\psi}_i^o (\hbar \omega_p + \hat{\mathbb{M}}_i) \psi_i^o dV \\ &= \hbar \omega_p \int \bar{\psi}_i^o \psi_i^o dV + \int \bar{\psi}_i^o \hat{\mathbb{M}}_i \psi_i^o dV \end{aligned}$$

$$= \hbar \omega_p + i \hbar \int \bar{\psi}_i^o (\vec{w}_i \nabla \psi_i^o - \frac{1}{2} \psi_i^o \nabla \cdot \vec{w}_i) dV$$

$$= \hbar \omega_p - i \frac{\hbar}{2} \int \nabla \cdot (|\psi_i^o|^2 \vec{w}_i) dV$$

$= \hbar \omega_p$ from **theorem of Gauss** and $\psi_i^o = 0$ out of particle's small volume

$$= H_i \quad \text{from (21),}$$

and hence, we obtained the equation (65) expressing the relationship

between the Hilbert space of the Hamiltonian operator $\hat{H}$ and the n different Hilbert spaces of the individual particle's IQM fundamental operators $\hat{\mathbb{F}}_i$ of corresponding classical ontological wavefunctions ψ_i^o , for $1 \leq i \leq n$.

□

Note that the physical meaning of statistical theory of SQM, based on unique Hamiltonian operator $\hat{H}$, used in equation (65), is explicitly provided by the equation (67).

Moreover, the equation (66) explains why in such a quantization of $\hat{\mathbb{F}}_i$ we can forget its hidden variables to obtain the non-hidden-variables statistical QM theory with the unique Hamiltonian operator $\hat{H}$: this phenomena is explained by application of theorem of Gauss in computation of $\langle \psi_i^o | \hat{\mathbb{F}}_i | \psi_i^o \rangle$.

Emergence of Born Rule from Ontological Ensemble

To complete this work, we will define the *ontological ensemble* of a massive particles, and the relationship with the quantum ensemble (41). Differently from a *quantum ensemble* of similarly prepared measurement systems, the *ontological ensemble* considers $N \gg 1$ repeated measurements of the massive elementary particle in the conditions specified by quantum ensemble. Thus, we have the strong relationship between these two kinds of ensemble, with the common setting: the "similarly prepared measurement systems".

So, ontological ensemble is defined by considering the massive particle with the rest mass m_0 , with ontological wave functions ψ_i^o (such that $|\psi_i^o|^2$ is not probabilistic density but a normalized energy-density) and its corresponding fundamental classic operator $\hat{\mathbb{F}}_i$ for $1 \leq i \leq n$, during the set of repeated measurements with identically prepared system for each of them (as required by statistical quantum EI ensemble).

Thus, in what follows we assume that, as in Corollary 1, the eigenfunction e_i in o.n.b. of the Hilbert space $\mathcal{H}$ of the Hamiltonian

operator $\hat{H}$ and, from the ontological point of view, the real particle which participate to the measurement of this individual system is represented by ontological wavefunction $\psi_i^o \in \mathcal{B}_p(\Delta t)$.

So, in our ontological layer of QM we obtained ontological representation of massive individual particle by its classical ontological complex wavefunction ψ_i^o in (53), with real normalized density $\rho_i(t, \vec{r}) = |\psi_i^o|^2$, i.e. equal to $\frac{\Phi_m}{m_0 c^2}$ and its density flow $\vec{J}_i = \rho_i(t, \vec{r}) \vec{w}_i(t, \vec{r})$ with classical continuity equation, derived by dividing the equation (24) with particle's rest-mass energy $m_0 c^2$, for each $1 \leq i \leq n$,

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot \vec{J}_i = 0 \quad (68)$$

which is a system that is:

- Fully deterministic
- Non linear (in general)
- Defined in 3D space

Thus, we can define the following energy multi-particle operator $\hat{\mathbb{F}}_i$ which can be applied only to one ontological wavefunction, can be *applied to every ontological wavefunction*, like the Hamiltonian operator for Schrödinger's wavefunctions:

Proposition 3 Given a n -particle system, $n \geq 1$, with orthonormal vector base $\mathcal{B}_p(\Delta t)$ defined by (51), we define the following functions:

1. The vector-decomposition function D_n that reduce a vector

$$\sum_{i=1}^k c_i \psi_i^o, \quad 1 \leq k \leq n, \quad \text{with base orthonormal vectors}$$

$$\psi_i^o = \frac{\Phi_i}{\sqrt{m_i c^2}} e^{-i\varphi_{T_i}} \in \mathcal{B}_p(\Delta t) \quad \text{and constants } c_i \in \mathbb{C},$$

into the n -tuple $[c_1 \psi_1^o \dots c_k \psi_k^o]$ of its mutually-orthogonal vector components, such that for $k = 1$ it is the identity and for $k \geq 2$,

$$D_n(\sum_{i=1}^k c_i \psi_i^o) = [c_1 \psi_1^o \dots c_n \psi_n^o],$$

2. The function id_T which is an identity for $k = 1$, and for $k \geq 2$ transforms a n -tuple row in the identical n -tuple column, that is

$$id_T([c_1 \psi_1^o \dots c_k \psi_k^o]) = [c_1 \psi_1^o \dots c_n \psi_n^o]^T = \begin{pmatrix} c_1 \psi_1^o \\ \vdots \\ c_k \psi_k^o \end{pmatrix}$$

3. The n -tuple phase-projection function π^n , such that for $k \geq 2$,

$$\pi^n([c_1 \psi_1^o \dots c_k \psi_k^o]) = [e^{-i\varphi_{T_1}} \dots e^{-i\varphi_{T_k}}]$$

while for $k = 1$, $\pi^n(c_1 \psi_1^o) = e^{-i\varphi_{T_1}}$.

4. The n -tuple generalization of natural logarithm ln_n , such that for $k \geq 2$,

$$ln_n([e^{-i\varphi_{T_1}} \dots e^{-i\varphi_{T_k}}]) = [-i\varphi_{T_1} \dots i\varphi_{T_k}]$$

while, for $k = 1$, reduces to the standard one-argument natural logarithm ln .

5. The n -tuple generalization of partial derivation $\frac{\partial_n}{\partial t}$, such that for $k \geq 2$,

$$\frac{\partial_n}{\partial t}([-i\varphi_{T_1} \dots i\varphi_{T_k}]) = [\frac{\partial}{\partial t}(-i\varphi_{T_1}) \dots \frac{\partial}{\partial t}(-i\varphi_{T_k})]$$

while, for $k = 1$, reduces to the standard one-argument natural logarithm $\frac{\partial}{\partial t}$.

6. The binary multiplication $\times(_, _)$, standard for $k = 1$ and matrix multiplication for $k \geq 2$.

and hence for the orthonormal vector base $\mathcal{B}_p(\Delta t)$, the following IQM operator is linear and Hermitian (self-adjoint) with time variable introduced by partial derivation operations,

$$\hat{\mathbb{H}}_n^o \equiv i\hbar \times (\frac{\partial_n}{\partial t} \circ ln_n \circ \pi^n, id_T) \circ D_n \quad (69)$$

where id_T is the tuple-identity function and $\circ$ denotes the composition of functions.

Proof: Thus, from definition of the operator $\hat{\mathbb{H}}_n^o$, for each single $\psi_i^o \in \mathcal{B}_p(\Delta t)$ and $c \in \mathbb{C}$, when $k = 1$, we obtain

$$\begin{aligned} \hat{\mathbb{H}}_n^o c \psi_i^o &= i\hbar \times (\frac{\partial_n}{\partial t} \circ ln_n \circ \pi^n, id_T) \circ D_n(c \psi_i^o) \\ &= i\hbar \times (\frac{\partial_n}{\partial t} \circ ln_n \circ \pi^n, id_T)(c \psi_i^o) = i\hbar \times (\frac{\partial_n}{\partial t} \circ ln_n \circ \pi^n(c \psi_i^o), id_T(c \psi_i^o)) \end{aligned}$$

$$\begin{aligned} i\hbar \times \left(\frac{\partial_n}{\partial t} \circ \ln_n(e^{-i\varphi_{T_i}}), c\psi_i^o \right) &= i\hbar \times \left(\frac{\partial_n}{\partial t}(-i\varphi_{T_i}), c\psi_i^o \right) \\ &= c\hbar \frac{\partial \varphi_{T_i}}{\partial t} \psi_i^o \\ &= cH_i \psi_i^o \quad \text{from (21),} \end{aligned}$$

so that for constant $c = 1$, we obtain that $\psi_i^o \in \mathcal{B}_P(\Delta t)$ are indeed its eigenfunction, that is, from the fact that $\hbar \frac{\partial \varphi_{T_i}}{\partial t} = H_i$ is the total energy (Hamiltonian) of i-th eigenfunction (i.e., i-th particle),

$$\hat{\mathbb{H}}_n \psi_i^o = H_i \psi_i^o \quad \text{and} \quad \overline{\hat{\mathbb{H}}_n \psi_i^o} = \overline{H_i \psi_i^o} \quad (70)$$

Thus, the Hamiltonian H_i of i-th particle is the eigenvalue of this operator $\hat{\mathbb{H}}_n$, for its eigenfunction ψ_i^o , in this n-dimensional Hilbert space $\mathcal{H}_P$ with the inner product such that $\langle \psi_k^o | \psi_k^o \rangle \equiv \int \overline{\psi_k^o} \psi_k^o dV = 1$, and

$$\langle \psi_i^o | \hat{\mathbb{H}}_n \psi_i^o \rangle = \int \overline{\psi_i^o} \hat{\mathbb{H}}_n \psi_i^o dV = H_i \quad (71)$$

equal to the total energy of i-th individual particle.

Moreover, for each constant $c \in \mathbb{C}$, from above, we obtain the following property of this operator:

$$\hat{\mathbb{H}}_n(c\psi_i^o) = c\hat{\mathbb{H}}_n\psi_i^o \quad (72)$$

Let us show that this operator is Hermitian (self-adjoint). For any two its eigenvalues ψ_i^o, ψ_j^o we have one of the two cases:

1. for the case when $i \neq j$:

$$\begin{aligned} \langle \psi_i^o | \hat{\mathbb{H}}_n \psi_j^o \rangle &\equiv \int \overline{\psi_i^o} \hat{\mathbb{H}}_n \psi_j^o dV \\ &= \int \overline{\psi_i^o} H_j \psi_j^o dV \quad \text{from (70)} \\ &= H_j \langle \psi_i^o | \psi_j^o \rangle = 0 = H_i \langle \psi_i^o | \psi_j^o \rangle \\ &= \langle H_i \psi_i^o | \psi_j^o \rangle \\ &= \int \overline{H_i \psi_i^o} \psi_j^o dV \\ &= \int \overline{\hat{\mathbb{H}}_n \psi_i^o} \psi_j^o dV \quad \text{from (70)} \\ &= \langle \hat{\mathbb{H}}_n \psi_i^o | \psi_j^o \rangle \end{aligned}$$

2. Case when $j = i$:

$$\langle \psi_i^o | \hat{\mathbb{H}}_n \psi_i^o \rangle \equiv \int \overline{\psi_i^o} \hat{\mathbb{H}}_n \psi_i^o dV = \int \overline{\psi_i^o} H_i \psi_i^o dV$$

from (70)

$$\begin{aligned} &= \int \overline{H_i \psi_i^o} \psi_i^o dV \\ &= \int \overline{\hat{\mathbb{H}}_n \psi_i^o} \psi_i^o dV \quad \text{from (70)} \\ &= \langle \hat{\mathbb{H}}_n \psi_i^o | \psi_i^o \rangle \end{aligned}$$

Thus $\hat{\mathbb{H}}_n$ is self-adjoint operator.

Let us show that it is linear as well, for example, for $k = 2$, taking two different eigenvectors $\psi_1^o = \rho_1 e^{-i\varphi_{T_1}}$ and $\psi_2^o = \rho_2 e^{-i\varphi_{T_2}}$ we can define their linear composition

$$\psi^o = c_1 \psi_1^o + c_2 \psi_2^o, \quad c_1, c_2 \in \mathbb{C} \quad (73)$$

and to apply this operator $\hat{\mathbb{H}}_n$, to it,

$$\begin{aligned} \hat{\mathbb{H}}_n \psi^o &= \hat{\mathbb{H}}_n(c_1 \psi_1^o + c_2 \psi_2^o) \\ &= i\hbar \times \left(\frac{\partial_n}{\partial t} \circ \ln_n \circ \pi^n, id_T \right) \circ D_n(c_1 \psi_1^o + c_2 \psi_2^o) \\ &= i\hbar \times \left(\frac{\partial_n}{\partial t} \circ \ln_n \circ \pi^n, id_T \right) ([c_1 \psi_1^o \ c_2 \psi_2^o]) \\ &= i\hbar \times \left(\frac{\partial_n}{\partial t} \circ \ln_n \circ \pi^n ([c_1 \psi_1^o \ c_2 \psi_2^o]), id_T ([c_1 \psi_1^o \ c_2 \psi_2^o]) \right) \\ &= i\hbar \times \left(\frac{\partial_n}{\partial t} \circ \ln_n \circ \pi^n ([c_1 \psi_1^o \ c_2 \psi_2^o]), [c_1 \psi_1^o \ c_2 \psi_2^o]^T \right) \\ &= i\hbar \times \left(\frac{\partial_n}{\partial t} \circ \ln_n ([e^{-i\varphi_{T_1}} \ e^{-i\varphi_{T_2}}]), [c_1 \psi_1^o \ c_2 \psi_2^o]^T \right) \\ &= i\hbar \times \left(\frac{\partial_n}{\partial t} ([-i\varphi_{T_1} \ -i\varphi_{T_2}]), [c_1 \psi_1^o \ c_2 \psi_2^o]^T \right) \\ &= i\hbar \times \left([-i \frac{\partial}{\partial t} \varphi_{T_1} - i \frac{\partial}{\partial t} \varphi_{T_2}], [c_1 \psi_1^o \ c_2 \psi_2^o]^T \right) \\ &= i\hbar \left[-i \frac{\partial}{\partial t} \varphi_{T_1} - i \frac{\partial}{\partial t} \varphi_{T_2} \right] \begin{pmatrix} c_1 \psi_1^o \\ c_2 \psi_2^o \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 &= \left[\hbar \frac{\partial}{\partial t} \varphi_{T_1} \hbar \frac{\partial}{\partial t} \varphi_{T_2} \right] \begin{pmatrix} c_1 \psi_1^o \\ c_2 \psi_2^o \end{pmatrix} \\
 &= c_1 \hbar \frac{\partial \varphi_{T_1}}{\partial t} \psi_1^o + c_2 \hbar \frac{\partial \varphi_{T_2}}{\partial t} \psi_2^o \\
 &= c_1 H_1 \psi_1^o + c_2 H_2 \psi_2^o \quad \text{from (21)} \\
 &= c_1 \hat{\mathbb{H}}_n \psi_1^o + c_2 \hat{\mathbb{H}}_n \psi_2^o \quad \text{from (70).} \\
 &\square
 \end{aligned}$$

In the approach codified by John von Neumann, a measurement of a physical system is represented by a self-adjoint linear operator on the Hilbert space. The eigenvectors of such an operator form an orthonormal basis for the Hilbert space, and each possible outcome of that measurement corresponds to one of the vectors comprising the basis. Thus, a measurement of a physical system is represented by this linear self-adjoint operator $\hat{\mathbb{H}}_n$ on the Hilbert space denoted by $\mathcal{H}_p$. These facts strongly constrain the underlying deterministic dynamics which is not imposed because:

1. $|\psi_i^o|^2 = \rho_i$ already has normalization $\langle \psi_i^o | \psi_i^o \rangle = 1$, as provided in (59),
2. Conservation of $\int \rho dV$, so that:
 - norm preservation
 - deterministic wavefunction evolution (61)
 - $\mathcal{H}_p$ Hilbert structure forced

So, this "ontological" Hilbert space $\mathcal{H}_p$ with orthonormal base

equal to the eigenfunctions ψ_i^o of the IQM classical operator $\hat{\mathbb{H}}_n$, is dual to the "statistical" Hilbert space with orthonormal base equal to the eigenfunctions e_i of the SQM quantum Hamiltonian operator $\hat{H}$. However, both Hilbert spaces have the same set of eigenvalues.

That is, we obtained the following strong relationship between the eigenfunctions e_i of the Hamiltonian operator $\hat{H}$ of a quantum ensemble, and the eigenfunctions ψ_i^o of the operator $\hat{\mathbb{H}}_n$, by using equation (66),

$$\langle e_i | \hat{H} | e_i \rangle = \langle \psi_i^o | \hat{\mathbb{H}}_n | \psi_i^o \rangle = H_i$$

by considering that both operators, classical $\hat{\mathbb{H}}_n$ and statistical Hamiltonian operator $\hat{H}$ are without hidden-variables.

Emergence of Born rule:

We recall that the normalized ontological particle's rest-mass energy-density $\rho_i(t, \vec{r})$ and its speed $\vec{w}_i(t, \vec{r})$ inside particle's finite volume are not observable, that is, the $\vec{J}_i = \rho_i \vec{w}_i$ is *hidden* flow which changes deterministically, caused by external fields that interact with massive particles.

Note, as provided previously in the process of quantization of the fundamental ontological equation in order to obtain the statistical Schrödinger equation, the hidden variables appear only in the ontological representation of massive individual particles and not in the SQM (statistical QM), so there is no sense to consider hidden variables in the statistical QM. The hidden variables are necessary components of the ontological *deterministic theory* of particles. So,

1. Measurement is coarse-graining of hidden flow and assume that:

- the underlying dynamics is deterministic
- but measurement only accesses coarse-grained configurations.

Thus, probabilities arise from ignorance over coarse-grained flow configurations.

2. Uniqueness of quadratic forms now impose:

- Probabilities depend only on the ontological ensemble ψ_e .
- Additivity over mutually exclusive outcomes.

The statistical *ensemble interpretation* (EI) makes distinction between Schrödinger complex wavefunctions and the physical entities involved. The physical entities are, for example, the electrons (an electron occupies a finite region of space which can not, at the same time, be occupied by another massive particle), but the wavefunctions are abstract (non-physical) mathematical concepts characterising probabilistically the positions of the particles just as the action in classical mechanics is a function characterizing the classical paths of particles.

From the fact that in the IQM theory we have no intrinsic quantum probability, but only the classical statistical probability of outcomes of an ensemble measurements (the mutual interference between particles does not exist because of the timespace separation (46) of energy-densities of particles, differently from eigenfunctions of the

Hamiltonian operator $\hat{H}$ that are, for each instant of time, present practically in whole 3-D space, so that we have the superposition of them), we do not need to use the particular mixed state operators (41) used for quantum ensemble. Ontological ensemble: So, we can define an *ontological ensemble* as a linear composition of eigenvectors in $\mathcal{B}_p(\Delta t)$, based on the property (73) of this $L^2(\mathbb{R}^3)$ Hilbert space $\mathcal{H}_p$, during whole process of repeated measurements. That is, for $t \in \Delta T$

$$\psi_e = \sum_{i=1}^n b_i \psi_i^o(t, \vec{r}), \quad b_i \in \mathbb{C}, \quad \psi_i^o \in \mathcal{B}_p(\Delta t) \quad (74)$$

is an ontological counterpart to quantum ensemble in (41), with the condition that we use very high number of repetitive measurements $N \gg n$, so that the i -th outcome H_i is obtained N_i times of the repetitive measurements, in the way that $\sum_{i=1}^n N_i = N$, and $\frac{N_i}{N}$ represents the well defined statistical probability that, for any single measurement, we will obtain the H_i output. So, in such well defined statistical framework of repetitive measurements, the ontological ensemble Ψ_e in (74) has to satisfy the condition for each $1 \leq i \leq n$,

$$\bar{b}_i b_i = \frac{N_i}{N} = P_i \quad \text{and hence} \quad \sum_{i=1}^n \bar{b}_i b_i = \sum_{i=1}^n P_i = 1 \quad (75)$$

where P_i is the probability to obtain H_i as outcome of a measurement. The Born rule provides a link between the mathematical formalism of quantum theory and experiment, and as such is almost single-handedly responsible for practically all predictions of statistical QM (SQM). Both Born and Heisenberg acknowledge the profound influence of Einstein on the probabilistic formulation of quantum mechanics. Whereas Born and the others just listed after him believed the outcome of any individual quantum measurement to be unpredictable in principle, Einstein felt this unpredictability was just caused by the incompleteness of quantum mechanics.

In effect, the development of IQM theory of massive non point-like individual particles with small volume of their rest-mass energy-density which generates internal energy-density flow, which is a hidden (non observable) variable of this theory, confirms Einstein's decision about incompleteness of standard statistical QM.

When a measurement device couples to the particle with energy-density flow, the measurement outcome depends on projection onto certain eigenvectors. Thus, this process provides the following generation of probabilities in an ontological ensemble of repetitive measurements Ψ_e for each i -th individual measurement, that is, the Born rule for ontological ensemble:

Corollary 2 Emergence of Born rule: *The Born rule is valid for the ontological ensemble as well, that is, the probability that the outcome of the eigenvalue H_i is given by⁶*

⁶The ontology ensemble Ψ_e in IQM theory is not SQM mixed state, so we do not need to use the mixed state operator ρ in (41).

$$P_i = \langle \Psi_e | \prod_i \Psi_e \rangle \quad (76)$$

where $\prod_i$ is Born's projection operator onto the basic eigenvector in $\mathcal{B}_p(\Delta t)$ to the measurement outcome H_i .

Proof: In fact, from (75) we have that

$$P_i = \frac{N_i}{N} = \bar{b}_i b_i$$

$$\begin{aligned} &= \bar{b}_i b_i \langle \Psi_i^o | \Psi_i^o \rangle \text{ from } \langle \Psi_i^o | \Psi_i^o \rangle = 1 \\ &= \langle b_i \Psi_i^o | b_i \Psi_i^o \rangle = \langle \sum_{k=1}^n b_k \Psi_k^o | b_i \Psi_i^o \rangle \quad \text{from the} \\ &\text{orthogonality of eigenvectors} \\ &= \langle \Psi_e | b_i \Psi_i^o \rangle \\ &= \langle \Psi_e | \prod_i \Psi_e \rangle \\ &\square \end{aligned}$$

So, let us show in details how the probabilities appear during the measurement process. From the fact that for the orthonormal eigenfunctions of the operator $\hat{\mathbb{H}}_n$, $\int \overline{\Psi_k^o} \Psi_i^o dV = \delta_{ik} = 1$ if $i = k$; 0 otherwise (δ_{ik} is Kronecker delta function), we obtain

$$\begin{aligned} \langle \Psi_e | \hat{\mathbb{H}}_n | \Psi_e \rangle &= \langle \Psi_e | \hat{\mathbb{H}}_n (\sum_{i=1}^n b_i \Psi_i^o) \rangle \\ &= \langle \Psi_e | \sum_{i=1}^n b_i \hat{\mathbb{H}}_n \Psi_i^o \rangle \quad \text{from linear operator } \hat{\mathbb{H}}_n \text{ property} \\ &= \langle \sum_{k=1}^n b_k \Psi_k^o | \sum_{i=1}^n b_i H_i \Psi_i^o \rangle \\ &= \int (\sum_{k=1}^n b_k \Psi_k^o) \sum_{i=1}^n b_i H_i \Psi_i^o dV \\ &= \sum_{i,k=1}^n \bar{b}_k b_i H_i \int \overline{\Psi_k^o} \Psi_i^o dV \\ &= \sum_{i,k=1}^n \bar{b}_k b_i H_i \delta_{ik} \quad \text{from orthogonality of eigenfunctions} \\ &= \sum_{i=1}^n |b_i|^2 H_i \\ &= \sum_{i=1}^n P_i H_i \text{ from (75)} \end{aligned} \quad (49)$$

which is the expectation value.

In order to have that the process of repeated n measurements of individual particle with rest-mass m_0 generates the same outcomes for the ensemble in both points of view, for ontological ensemble Ψ_e in (74) and for quantum ensemble represented by mixed state operators in (41), we obtain as expected that

$$(\hat{\mathbb{H}}_n)_{av} \equiv \langle \Psi_e | \hat{\mathbb{H}}_n | \Psi_e \rangle = (\hat{H})_{av}$$

Consequently, we derived the *Born rule* also inside the IQM theory of real individual massive particles.

So, we obtained the strong completion of Quantum Mechanics where the statistical part (SQM) is derived from the ontological theory that describes the real individual particles by using also hidden (non observable) variables, and also the probabilistic nature of Schrödinger wavefunctions:

- Determinism exists at the level of individual particle dynamics.
- Statistical QM still emerges for ensembles.
- Probabilities arise from incomplete knowledge of initial conditions. So the probabilistic structure is retained phenomenologically, but grounded in deterministic micro-dynamics.

This derivation of the statistical QM from the IQM theory of an individual massive particle with internal energy-density speed $\vec{w}$ as hidden variable. can be formulated by the following quantization transformations table:

quantization $\mapsto$	from classical IQM	to statistical QM
operators	$\hat{\mathbb{H}}_n^o = \bigcup_{i=1}^n \hat{\mathbb{D}}\mathcal{V}_i(t, \vec{r})$	$\hat{H}$
eigenfunctions	$\psi_i^o(t, \vec{r})$	$e_i(t, \vec{r})$
eigenvalues	Hamiltonian H_i	Hamiltonian H_i
ensemble	$\psi_e = \sum_{i=1}^n (e^{i\beta_i} \sqrt{P_i}) \psi_i^o$ $\sum_{i=1}^n P_i = 1$ for $\beta_i \in \mathbb{R}$	$\hat{\rho} = \sum_{i=1}^n P_i e_i\rangle \langle e_i $
Born rule	$P_i = \langle \psi_i^o \psi_e \rangle ^2$	$P_i = \langle e_i \psi \rangle ^2$
probabilistic average	$(\hat{\mathbb{H}}_n^o)_{av} = \sum_{i=1}^n P_i H_i$	$(\hat{H})_{av} = \sum_{i=1}^n P_i H_i$
PDE	$i\hbar \frac{\partial \psi_i^o}{\partial t} = \hat{\mathbb{F}}_i \psi_i^o$ ontological	$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$ Schrödinger

With this derivation of Schrödinger equation from the ontological equation of an individual massive elementary particle we obtain also the fundamental results of the SQM theory: For Hilbert space to emerge from timespace, timespace must contain

1. A structure that behaves like a complex vector space.
2. A natural inner product structure.
3. A linear composition of eigenvectors.
4. A reason for deterministic wavefunction evolution (this evolution preserves norm).

5. A geometric origin of the Born rule. The statistical Hilbert layer (Born rule ensemble) follows from the ontological layer.

The structure of operators $\hat{\mathbb{F}}_i$ of individual particles in (??), containing also hidden variables, are more expressive than no-hidden variables Hamiltonian structure, and Schrödinger equation emerges as statistical coarse-graining of ontological PDE of a massive individual particle. Consequently this is similar in spirit to hydrodynamics emerging from molecular dynamics.

With this, all statistical results of SQM for quantum ensemble are the consequences and can be physically explained by the IQM ontological theory of individual particles (which is a local hidden-variables theory). The multi-particle systems generally (not only for ontological ensemble) have no superposition effects, because each massive particle with its small volume of energy-density is separated spatially from other particles, and hence the superposition effect in statistical QM based on non-local eigenfunctions of Hamiltonian operator has no any plausible physical interpretation.

Conclusion

Putting all together we have:

Ontology: for each massive particle deterministic timespace field $(\rho_i, \vec{J}_i)$

Encodings: particle's ontology wavefunction $\psi_i^o = \sqrt{\rho_i} e^{-i\varphi_T}$, so that Hilbert space $L^2(\mathbb{R}^3)$ emerges

Dynamics: constraint of linear evolution $i\hbar \frac{\partial}{\partial t} \psi_i^o = \hat{\mathbb{F}}_i \psi_i^o$

Probability: symmetry + consistency. Born rule emerges uniquely

Note that the phase component of ψ_i^o , from (19), represents the minimum-action principle for an massive individual particle during its movement in timespace (under actions of all principal natural forces on this individual particle that deterministically change particles internal flow of rest-mass energy-density).

Einstein in his four lectures delivered at Princeton University, May 1931, published by Princeton University Press, 1922, [26], in Section "Phenomenological Representation of Energy Tensor of Matter", *Hydrodynamical Equations*, pp 55-56, expressed his ideas as follows:

"We know that matter is built up of electrically charged particles, but we do not know the laws which govern the constitution of these particles. In treating mechanical problems, we are therefore obliged to make use of an inexact description of matter, which corresponds to that of classical mechanics. The density σ , of a material substance and the hydrodynamical pressures are the fundamental concepts upon which such a description is based.

Let σ_0 be the density of matter at a place, estimated with reference to a system of coordinates moving with the matter. Then σ_0 , the density at rest, is an invariant. If we think of the matter in arbitrary motion and neglect the pressures (particles of dust in vacuo, neglecting the size of the particles and the temperature), then the energy tensor will depend only upon the velocity components, u_v , and σ_0 ."

This Einstein intuition is confirmed by my calculus, with my initial idea that matter of a particle propagates in the locally-flat Minkowski time-space, as a complex ontological wavefunction Ψ , such that $\Phi_m(t, \vec{r}) \equiv \bar{\Psi}\Psi$ corresponds to the Einstein idea of matter-density, which amount l_ϕ is an invariant, while $\vec{w}(t, \vec{r})$ is the matter-density speed, with components w_ν , corresponding to Einstein's speed components u_ν .

Consequently, the material exposed in this paper is a coherent continuation of the Einstein point of view, and these two fundamental physical properties, the mater (rest-energy) density distribution Φ_m and its speed (flux) $\vec{w}$, which are not part of the Hamiltonian-based QM, are both the kinds of hidden, non-observable parameters, needed for the completion of the Quantum Mechanics. In this paper we have shown their role in quantization process from the classical IQM theory of massive particles into statistical Schrödinger equation.

Shortly after making his famous "God does not play dice" comment, Einstein attempted to formulate a deterministic counter proposal to quantum mechanics, presenting a paper at a meeting of the Academy of Sciences in Berlin, on 5 May 1927, titled "Bestimmt Schrödinger's Wellenmechanik die Bewegung eines Systems vollständig oder nur im Sinne der Statistik?" ("Does Schrödinger's wave mechanics determine the motion of a system completely or only in the statistical sense?") [27]. However, as the paper was being prepared for publication in the academy's journal, Einstein decided to withdraw it, possibly because he discovered that, contrary to his intention, his use of Schrödinger's field to guide localized particles allowed just the kind of non-local influences he intended to avoid [28].

In fact, he could not use the Quantum Mechanics in that time based on Schrödinger's equation, because in this (standard) QM the elementary particles has been considered as point-like objects, thus without any possibility to introduce the hidden variables in the particle's internal structure and, moreover, Schrödinger's wavefunctions could not represent any real physical density like energy of charge, but only the density of probability.

Because of that, the probabilistic outcomes from equally prepared measurement systems could not be explained based on any physical ontological phenomena, and only because of that the Heisenberg uncertainty principle (1927) could have erroneous interpretation that "this principle limits determinism in quantum mechanics, shifting the view from predictable trajectories to probabilistic wave functions" with extreme conclusion that *in the micro-scale there is an ontological chaos*, differently from the all experience in our every-day life and observations.

The only way to resolve this contradictions, has been to introduce a new non point-like model of particle in which a real particle always have some finite volume for its rest-mass energy density. It is just done by developing this IQM theory, based on intuition of Einstein that a particle is a kind of wave-pocket of energy concentrated in a small volume, to explain what are the hidden variables in this model and which is fundamental classical physical law that governs its time evolution (conservation law).

In the IQM theory, from the fact that the rest-mass of each individual particle *is invariant* (equal for each reference system), it has been natural to define an individual particle as a rest-mass energy-

density with internal energy-density flow as hidden (non observable) variable, governed by well known continuity equation.

The continuity equation is a mathematical expression of the conservation of mass in fluid dynamics, stating that mass flow rate remains constant throughout a closed system. Thus, this is a an *equation of statistical classic mechanics*. That is, in order to define an *individual* particle, we have to use a *statistical equation* for its hidden variables (energy-density flow inside particle's volume). Just with this, intuitively, we have a motivation for physical explanation of *different outcomes* of two repetitive measurements of the same type of particles in two equivalently prepared measurement systems, also when we suppose that the initial conditions for both particles are equal.

The trick is that in order to establish that *two different particles have the same initial conditions* (say, for initial time t_0), we can use only observable particle's variables as velocity, momentum and total energy. But it is not enough, because in this initial time they (with high probability) will have very different values of their hidden variables (that can not be observed in any case). So, also *if the time evolution is deterministic* (as is for example deterministic also for Schrödinger's wavefunction Ψ), for each of these two particles in the equivalently prepared measurement laboratory systems, we will obtain often two different outcomes.

The probabilistic results of measurements, also under equivalently prepared measurement systems and equal observably equal conditions, are determined by physical existence of hidden energy-density flows inside massive particles which can not be verified externally: thus, the probabilistic outcomes in a quantum ensemble is a physical (ontological) phenomena. This physical phenomena for massive particles explains the Heisenberg uncertainty, which could not be explained in standard QM (SQM) based on the point-like particles.

From the fact that the leading IQM equation, from which is derived also equation for ontological particle's wavefunction Ψ , is the classical *statistical continuity equation*, and from the fact that already it was demonstrated the derivation of Schrödinger's equation from the classical statistical Liouville equation (in Section 1), it can not be an completely unexpected result that we could derive Schrödinger's equation also from the fundamental ontological wavefunction IQM equation. The property of continuity equation for any individual massive particle has been decisive in order to eliminate (by application of Gauss theorem) the hidden variables during reduction of deterministic IQM theory to no-hidden variables statistical QM theory (during derivation of Schrödinger's equation).

The practical implications of the proposed framework by using IQM theory are provided in [23, 24], for formal derivation electromagnetic theory as a statistical result of actions of long and short-range photons, physical explanation of Aharonov-Bohm effects determined by short-range (massive) photons, Tesla scalar waves, etc... The completion of standard quantum mechanical (SQM) formulations with deterministic IQM theory of individual particles, from the fact that now it is clear that Schrödinger's wavefunctions can not be used for individual particles, all current theories that use them in this way, as Bell's theorem, entanglement definition of two individual particles, etc..., has to be revisited and enriched by new knowledge provided by this competed quantum theory. This requires more future work by different groups of scientists specialized in these questions, which would strengthen the scientific relevance of the quantum theory and generate new discoveries.

The de Broglie-Bohm pilot-wave theory which is previously defined as a hidden-variable theory, is not applicable to individual particles (and their hidden-variables of internal energy-density flow), because its wavefunctions are plane waves with free variable of position $\mathbf{r}$ (for each instance of time present at any point of Universe) as demonstrated in [23, 24], so it can be eventually a part of statistical theory SQM as is Schrödinger's equation (derivable from the IQM theory) and its *probability-density* wavefunctions. However, it is demonstrated that it is not well defined theory because creates contradiction with special relativity (while the individual particles in the IQM theory are well defined in Minkowski time space and satisfy the special relativity as well). But special relativity is well found theory and experimentally confirmed. Only if would be demonstrated without any doubt that the special relativity does not hold, and that there is no limit speed for massive particles, this theory can be reconsidered.e:

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