

Submitted : 07 May, 2026

Accepted : 15 May, 2026

Published : 16 May, 2026

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Keywords: Wave equation; Vibration transport solutions; Green's function; Fourier transform; Helmholtz equation; Doppler effect

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Review Article

Subsonic Vibrotransport Solutions of D'Alembert Equation in Spaces of Dimensions $N = 3, 2$, and 1

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Abstract

Among active disturbance sources in various environments, the most common are transport and vibrotransport phenomena, which are associated with moving objects, whose speeds can be subsonic, sonic, supersonic, and in environments with several sonic speeds (elastic, for example) also transonic. Here, fundamental and regular vibrotransport solutions of the wave equation are constructed at subsonic speeds of the disturbance source in spaces of physical dimensions ($N = 3, 2, 1$). Green's functions are constructed to describe the dynamics of the medium during the movement of a source concentrated at a point, moving at constant speed and vibrating at constant frequency. Based on these results, general solutions of the vibration transport equation are constructed under the action of both spatially distributed moving vibration sources and concentrated on moving surfaces and lines. A mathematical description of the Doppler effect with a graphical illustration is given.

The constructed solutions allow one to construct solutions of many equations of continuum mechanics for this type of moving sources of disturbances in media and can be applied extensively in solving various engineering and technical problems.

Introduction

Among the active sources of disturbances in various media, transport ones are most common, which are associated with moving sources (loads), the shape of which does not change over time, and the speed of movement can be subsonic, sonic, supersonic, and in environments with several sound velocities (elastic, for example) also transonic. In [1-7], transport solutions of wave equations and equations of elasticity theory were constructed, and, based on the method of generalized functions, a method of boundary integral equations was developed for solving stationary subsonic and supersonic transport boundary value problems in areas with cylindrical boundaries. It should be noted that the number of works on the study of the impact of transport loads on the environment has been growing in recent decades due to the intensive construction of high-speed road and underground transport highways and has a fairly extensive bibliography, which can be found in articles and monographs [8-14].

There is another class of disturbance sources (acting forces and loads) that is very important for applications, which not only move at different speeds, but also pulsate (vibrate, oscillate) with a certain frequency. For example, electromagnetic emitters, moving elementary particles, mobile vibration transport. Therefore, mathematical modeling of such processes is essential, taking into account the type of source, the speed of its movement and the frequency of vibration. A class of such model problems is considered in this paper.

Fundamental solutions play a crucial role in the development of the Method of Fundamental Solutions (MOF) and the Method of Boundary Integral Equations (MBIE) for solving boundary value problems in mathematical physics equations. These solutions serve as the foundation for constructing the kernels of integral equations and integral representations of boundary value problem solutions.

Here, fundamental and regular vibration-transport solutions of the wave equation are developed for subsonic, supersonic, and sonic speeds of perturbation sources. Green's functions are constructed to describe the medium's dynamics when a point vibration source is moving. Based on these functions, general solutions to the vibration-transport equation are provided for both distributed moving vibration sources in space and sources concentrated on moving surfaces and lines.

The developed solutions enable the construction of solutions for many equations in continuum mechanics for this type of moving perturbation sources in various media and have extensive applications in solving various engineering and technical problems.

Statement of novelty

This study proposes a novel analytical framework for the construction of subsonic vibrotransport solutions of the D'Alembert wave equation in spaces of dimensions $N = 3, 2$ and 1 . Unlike classical models that deal separately with transport or vibration phenomena, this work simultaneously integrates the translational motion and the oscillatory behavior of the sources into a unified mathematical formulation. The derivation of Green's functions and generalized solutions for moving vibrating sources provides new insights into the mechanisms of wave propagation and extends existing methodologies in mathematical physics and continuum mechanics. This approach has potential for application in advanced engineering systems involving coupled transport and vibration phenomena.

1. The D'Alembert wave equation and its properties

A multidimensional analogue of the D'Alembert equation is considered.

$$\epsilon_c u \equiv \Delta u - c^{-2} \frac{\partial^2 u}{\partial t^2} = g(x, t), \quad x \in R^N, \quad t \in R^1 \quad (1)$$

Here ϵ_c - the wave operator (d'Alembertian), $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ - the Laplace operator, $g(x, t)$ is a locally integrated function.

Equation (1.1) is strictly hyperbolic, the class of its solutions contains functions that are discontinuous in derivatives [15]. The discontinuity surfaces in $R^{N+1}(F)$ are the characteristic surfaces of equation (1) that satisfy the characteristic equation in space $R^{N+1} = \{(x, \tau = ct)\}$:

$$\dot{\tau}^2 = \sum_{j=1}^N \dot{x}_j^2. \quad (2)$$

where is the $\nu(x, \tau) = (\nu_1, \dots, \nu_N, \nu_\tau)$ normal vector to F , $\tau = ct$. It corresponds to a cone of characteristic normals - a light cone for which $\nu_\tau = \nu_{N+1} < 0$ [1,2]. In R^N such surfaces, they move with a single velocity along τ :

$$1 = -\dot{\tau} / \|\nu\|_N, \quad \|\nu\|_N = \sqrt{\nu_j \nu_j} \quad (3)$$

(according to the repeated indices i, j in the product, here and further, summation from 1 to N is carried out everywhere). In space R^N they correspond to wave fronts (F_t) moving at a speed c in time t . The Hadamard continuity conditions are fulfilled on them:

$$[u(x, t)]_{F_t} = 0, \quad [\dot{u}]_{F_t} = -cn_i [u, i]_{F_t} \quad (4)$$

where through $[f(x, t)]_{F_t}$ marked by a jump of f to F_t :

$$[f(x, t)]_{F_t} = f^+(x, t) - f^-(x, t) = \lim_{\varepsilon \rightarrow +0} (f(x + \varepsilon n, t) - f(x - \varepsilon n, t)), \quad x \in F_t,$$

$n(x, t)$ is a unit vector of the normal to F_t , directed towards the propagation of the wave front:

$$n_i = \frac{\nu_i}{\|\nu\|_N} = \frac{\text{grad } F_t}{\|\text{grad } F_t\|}, \quad i = 1, \dots, N. \quad (5)$$

The latter equality is valid if the equation of the wave front can be represented in the form $F_t(x, t) = 0$ under the condition of existence $\text{grad } F_t$.

A class of such solutions to hyperbolic equations is called shock waves, on their fronts the derivatives of functions and even the functions themselves can suffer jumps. From the second condition (1.4) it follows that on the fronts

$$\dot{u}^- + cn_i u_{,i}^- = \dot{u}^+ + cn_i u_{,i}^+ \quad (6)$$

If in front of the wave front $u \equiv 0$ (the medium is at rest), this equality gives a useful ratio at the wave front:

$$(grad u, n) = -c^{-1} \dot{u}, \quad x \in F_t$$

Note that the tangent derivatives to the characteristic surface, due to the continuity of u , are also continuous, i.e.,

$$\gamma_\tau [u_{,\tau}]_F = -\gamma_j [u_{,j}]_F \text{ for } \forall \gamma : (v, \gamma) = 0. \quad (7)$$

In particular, if $\gamma = \gamma^j = (-v_j, v_\tau \delta_1^j, v_\tau \delta_2^j, v_\tau \delta_3^j)$, this leads to conditions of the form:

$$[-u_{,\tau} v_j + u_{,j} v_\tau]_F = 0 \Rightarrow n_j [\dot{u}]_{F_t} = c [u_{,j}]_{F_t} \quad (8)$$

The solutions of the equation of the wave equation (1.1) satisfying the conditions at the shock wave fronts are further called classical.

2. Formulation of the vibrotransport problem

Definition 1. We define the source function $g(x, t)$ vibro-transport if it is represented as

$$g(x, t) = g(x_1, x_2, x_3 - Vt) e^{i\omega t} \quad (9)$$

Where, V is the velocity of the source along the X_3 axis, ω is the frequency of its oscillations, $\omega > 0$. The load is assumed to be transportable.

If the right-hand side of the wave equation (1) has the form (9), then it is natural to look for a solution in this form:

$$u(x, t) = u(x_1, x_2, x_3 - Vt) e^{i\omega t} \quad (10)$$

To do this, let's switch to a movable coordinate system $(x_1, x_2, z = x - M\tau), \tau = ct, M = V/c$ - the Mach number. We define the source subsonic if $M < 1$, supersonic if $M > 1$, and sonic if $M = 1$.

In the new coordinate system, the solution looks like:

$$u = u(x_1, x_2, z) e^{i w \tau}, \quad w = \omega / c$$

Then, as follows from (1), the amplitude of the oscillations is the solution of the Vibration Transport Equation (VTE):

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + (1 - M^2) \frac{\partial^2 u}{\partial z^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^2, z \in R^1 \quad (11)$$

Let us denote $m = \sqrt{|1 - M^2|}$. Then, depending on the velocity of the source, we have three different equations:

for $M < 1$ subsonic elliptic

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + m^2 \frac{\partial^2 u}{\partial z^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^2, z \in R^1; \quad (12)$$

for $M > 1$ supersonic hyperbolic

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - m^2 \frac{\partial^2 u}{\partial z^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^2, z \in R^1; \quad (13)$$

for $M=1$ sound parabolic

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^N, z \in R^1. \quad (14)$$

It is required to construct a solution of these equations for any right-hand sides from the class of generalized slow-growth $S'(R^3)$ functions [16,17].

3. Fundamental solutions of the vibration transport equation

Fourier transformation

To construct solutions to equation (14), we construct the Green's function - the fundamental solution $U(x, z)$ of this equation with the delta function on the right side:

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + (1 - M^2) \frac{\partial^2 U}{\partial z^2} + 2i w v \frac{\partial U}{\partial z} + w^2 U = \delta(x) \delta(z), \quad x \in R^N, z \in R^1 \quad (15)$$

Which satisfies specific decay conditions at infinity, which vary for each case. Using the properties of the Green's function, we construct solutions to the VTE for moving vibration sources, either distributed within bounded volumes or concentrated on curvilinear lines.

To construct solutions, we use the Fourier transform of generalized functions, which for summable regular generalized functions coincides with the classical Fourier transform:

$$\bar{f}(\xi, \zeta) = \int_R f(x, z) \exp(i(x_1 \xi_1 + x_2 \xi_2 + z \zeta)) dx_1 dx_2 dz \quad (16)$$

$$f(x, z) = \frac{1}{(2\pi)^3} \int_R \bar{f}(\xi, \zeta) \exp(-i(x_1 \xi_1 + x_2 \xi_2 + z \zeta)) d\xi_1 d\xi_2 d\zeta$$

From equation (15), we obtain

It follows that:

$$\text{for } M < 1 \quad \bar{U} = -\frac{1}{\|\xi\|^2 + m^2 \zeta^2 - 2wM\zeta - w^2}, \quad (17)$$

$$\text{for } M > 1 \quad \bar{U} = -\frac{1}{\|\xi\|^2 - m^2 \zeta^2 - 2wM\zeta - w^2}, \quad (18)$$

$$\text{for } M = 1 \quad \bar{U} = -\frac{1}{\|\xi\|^2 - 2w\zeta - w^2}, \quad (19)$$

In this article, we consider the subsonic case. The appearance of the original depends on the dimension of the space in which this equation is considered. Here we construct $U(x, z)$ for spaces of physical dimension $N=3, 2, 1$

4. Solutions of the vibration transport equation for the motion of regular and singular vibration sources in 3D space

4.1. Green's function, $N=3$

Let's construct the Green's function $U(x, z)$ is the fundamental solution of VTE (12), satisfying the radiation conditions at infinity. To do this, we will find the transformation $U(x, z) = F^{-1}[\bar{U}(\xi, \zeta)]$ using the property of linear transformations of coordinates in the space of Fourier transforms.

Lemma 1. For $N=3$

$$U(x, z) = -\frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{-i\zeta z} e^{-i(\xi, x)}}{\|\xi\|^2 + m^2 \zeta^2 - 2wM\zeta - w^2} d\xi_1 d\xi_2 d\zeta =$$

$$= -\frac{e^{-i(wMz/m^2)}}{(2\pi)^3 m} \int_{R^3} \frac{e^{-i\zeta z/m} e^{-i(\xi, x)}}{\|\xi\|^2 + \zeta^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta.$$

Proof: For $M < 1$, we transform (17) to a form convenient for constructing the original:

$$\bar{U}(\xi, \zeta) = -\frac{1}{\|\xi\|^2 + m^2 \zeta^2 - 2wM\zeta - w^2} = -\frac{1}{\|\xi\|^2 + m^2 \left(\zeta - wM/m^2\right)^2 - (w/m)^2} \quad (20)$$

$$U(x, z) = -\frac{1}{(2\pi)^N} \int_{R^3} \frac{e^{-i\zeta z} e^{-i(\xi, x)}}{\|\xi\|^2 + m^2 \left(\zeta - wM/m^2\right)^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta =$$

$$= -\frac{e^{-i(wMz/m^2)}}{(2\pi)^N} \int_{R^3} \frac{e^{-i\zeta z} e^{-i(\xi, x)}}{\|\xi\|^2 + m^2 \zeta^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta =$$

$$= -\frac{e^{-i(wMz/m^2)}}{(2\pi)^N m} \int_{R^3} \frac{e^{-i\zeta z/m} e^{-i(\xi, x)}}{\|\xi\|^2 + \zeta^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta \quad (21)$$

Variable substitution was used here $\zeta = m(\zeta + wM/m^2)$. Note that here, under the sign of the integral, there is a Fourier transform of the fundamental solution of the three-dimensional Helmholtz equation:

$$\Delta W + k^2 W = \delta(y), \quad k = \frac{w}{m}, \quad y \in R^3$$

The solution of this equation satisfying the Sommerfeld radiation conditions [15,16] has the following form:

$$W(y) = \frac{\exp(-ik\|y\|)}{4\pi\|y\|}, \quad y \in R^3 \quad (22)$$

Its Fourier transform has the form

$$\bar{W} = -\frac{1}{\|\xi\|^2 + \zeta^2 - (k + i0)^2}, \quad y \in R^3 \quad (23)$$

From formula (20), taking into account (22) and (23), it follows:

$$\text{for } N=3 \quad U(\mathbf{x}, z) = U(x_1, x_2, z) = -\frac{e^{-iwm^{-2}Mz}}{4\pi\sqrt{z^2 + m^2 r^2}} \exp\left(-\frac{iw}{m^2} \sqrt{z^2 + m^2 r^2}\right) \quad (24)$$

4.2. Solutions of homogeneous VTE for N=3

We now construct solutions of a homogeneous VTE:

$$\frac{\partial^2 u^0}{\partial x_1^2} + \frac{\partial^2 u^0}{\partial x_2^2} + 2i wM \frac{\partial u^0}{\partial z} + w^2 u^0 =, \quad x \in R^2, z \in R^1; \quad (25)$$

In the space of Fourier transforms, it has the form:

$$-\left(\|\xi\|^2 + (1 - M^2)\zeta^2 - 2wM\zeta - w^2\right)\bar{u}^0 = 0 \quad \xi \in R^N, \zeta \in R^1 \quad (26)$$

The solution to this equation $\bar{u}^0 = \alpha(\xi, \zeta)\delta_S(\xi, \zeta)$ is a singular generalized function - a simple layer on a surface S on which

$$\left(\|\xi\|^2 + (1 - M^2)\zeta^2 - 2wM\zeta - w^2\right) = \|\xi\|^2 + m^2\left(\zeta - wM/m^2\right)^2 - (w/m)^2 = 0. \quad (27)$$

Here, the density of a simple layer is an arbitrary function integrable on S .

Accordingly

$$u^0(x, z) = \int_S \alpha(\xi, \zeta) e^{-i(\xi, x)} e^{-i\zeta z} dS(\xi, \zeta), \quad \xi = (\xi_1, \xi_2) \quad (28)$$

Note that equation (28) is the equation of an ellipsoid centered at a point $(0, 0, \zeta = wM/m^2)$:

$$\|\xi\|^2 + m^2\zeta^2 = (w/m)^2, \quad \zeta = \zeta - wM/m^2. \quad (29)$$

Solutions of the homogeneous Helmholtz equation can also be used to construct $u^0(x, z)$:

$$\Delta u^0(y) + k^2 u^0(y) = 0. \quad (30)$$

Its solutions can be decomposed into series according to spherical harmonics and spherical Bessel functions [17,18]:

$$\begin{aligned} u(y) &= \sum_{n,m} a_n j_n(k\|y\|) P_n^m(\cos\theta) e^{im\varphi} = \sum_{n,m} a_n j_n(k\|y\|) P_n^m\left(\frac{y_3}{\|y\|}\right) (\cos\varphi + i\sin\varphi)^m = \\ &= \sum_{n,m} a_n \frac{j_n(k\|y\|)}{\|y\|_2^m} P_n^m\left(\frac{y_3}{\|y\|}\right) (y_1 + iy_2)^m, \quad \|y\|_2 = \sqrt{y_1^2 + y_2^2} \end{aligned} \quad (30)$$

Here $P_n^m(\cos\theta)$ are the attached Legendre polynomials, θ, φ angular spherical coordinates. It follows $y = (x, z/m)$ from formula (21)

$$u^0(x, z) = e^{-i(wMz/m^2)} \sum_{n,l} a_n j_n\left(\frac{w}{c}\sqrt{z^2 + m^2 r^2}\right) P_n^l\left(\frac{z}{\sqrt{z^2 + \mu^2 r^2}}\right) \frac{(x_1 + ix_2)^l}{r^l}, \quad (31)$$

Where $r = \sqrt{x_1^2 + x_2^2}$, the coefficients a_n are arbitrary complex numbers.

4.3. The general solution of the VTE for $N = 3$

Let's prove the following theorem.

Theorem 1. The solution of VTE (12) in 3D space has the following form:

$$u(x, z) = U(x, z) * g(x, z) + u^0(x, z). \quad (33)$$

If $g(x, z)$ is a regular function and $g(x, z) \in L_1(R^3)$, then

$$U(x, z) * g(x, z) = \int_{R^3} U(x - y, z - h) g(y, h) dy_1 dy_2 dh. \quad (34)$$

If $g(x, z)$ is a singular function centered on the surface S : $g(x, z) = \alpha(x, z)\delta_S(x, z)$, $\alpha(x, z) \in L_1(S)$, then

$$U(x, z) * g(x, z) = \int_S U(x - y, z - h) g(y, h) dS(y, h) \quad (35)$$

If $g(x, z)$ is a singular function centered on the curve l : $g(x, z) = \beta(x, z)\delta_l(x, z)$, $\beta(x, z) \in L_1(l)$, then

$$U(x, z) * g(x, z) = \int_l U(x - y, z - h)g(y, h)dl(y, h) \quad (36)$$

Proof. Let's denote the differential operator VTE $VT(\partial_1, \partial_2, \partial_z)$ (12). Substituting (33) into (12) we obtain the required:

$$\begin{aligned} & VT(\partial_1, \partial_2, \partial_z) \left(U(x, z) * g(x, z) + u^0(x, z) \right) = \\ & = \{ VT(\partial_1, \partial_2, \partial_z) U \} * g + VT(\partial_1, \partial_2, \partial_z) u^0 = \delta(x, z) * g + 0 = g(x, z) \end{aligned}$$

Here we used the linearity of the operator, (15), (25) and the convolution property with the delta function [16,17].

If $u_1(x, z)$ is any solution (12), then it $u_2(x, z) = u(x, z) - u_1(x, z)$ is a solution of a homogeneous VTE (25). Therefore $u_1(x, z) = u(x, z) - u_2(x, z)$. That is, it has a similar one $u_1(x, z)$.

4.5. The Doppler effect

Let's denote $r/z = \text{tg} \varphi(x, z)$, where φ is the angle that forms the radius vector of the point (x, z) with the Z axis. then the Green's function can be written as:

$$U(x, z) = -\frac{1}{4\pi\sqrt{z^2 + m^2r^2}} \exp\left(-i\alpha z \left(M + \sqrt{1 + m^2\text{tg}^2\varphi(x, z)}\right)\right)$$

As observed, a wave of the form is spreading along the X_3 axis

$$\varphi(x_1, x_2, x_3, t) = -\frac{1}{4\pi|x_3 - Vt|} \exp\left(i\omega \left(t - \frac{(M+1)|x_3 - Vt|}{cm^2}\right)\right).$$

If we fix the observation point (x_1, x_2, x_3) and measure the time-arriving signal at this point, then it is described by the function (Figure 1):

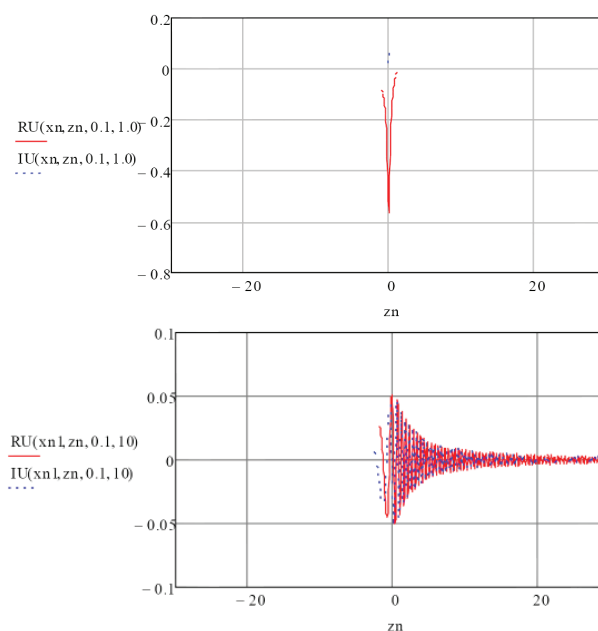


Figure 1: Real and imaginary parts of the vibrotransport wave field for a subsonic source (Mach number $M = 0.1$). The figure illustrates the spatial distribution of the real part $\text{Re}(U)$ and imaginary part $\text{Im}(U)$ of the solution, corresponding to a vibration source moving along the X_3 -axis with constant velocity V and oscillation frequency. The results demonstrate the modulation of wave amplitude and phase in the moving coordinate system, where frequency increases in the direction of motion due to Doppler-type effects. Parameters used include $M = 0.1$ and selected characteristic frequencies ω_0 and ω_1 .

Figure 1 shows the real (RU(...)) and imaginary (IU(...)) parts $U(x, z)$ for Mach number $M=0.1$ and frequencies $\omega = 1$ and $\omega = 10$. In a moving coordinate system, the frequency of oscillations increases in front of the moving source. But in the original fixed (x, x_2, x_3) coordinate system, the picture is different.

Figure 2 shows the waveform of the signal at a fixed point on the X_3 axis over time $t=t_n$ for Mach number $M=0.8$ and vibration frequency $\omega = 10$. This demonstrates an increase in the frequency and amplitude of vibration when approaching a vibration source and, conversely, their decrease when it is removed.

According to classical wave theory, the pressure in the air satisfies the wave equation [19]. This phenomenon is called the *Doppler effect* – an increase in tone (frequency) and volume (amplitude) when approaching a vibration source and, conversely, a decrease in tone and volume when it is removed.

The obtained results are consistent with classical Doppler theory, where the observed frequency depends on the relative motion between the source and the observer. The present formulation extends this concept by incorporating vibrotransport effects, thereby providing a more comprehensive description of wave behavior for moving oscillatory sources.

5. Solutions of the vibration transport equation for the motion of regular and singular vibration sources in 2D space

5.1. Green's function N=2

Let's construct the Green's function $U(x, z)$ similarly to the above. Its inverse Fourier transform in this case has the form:

$$U(x, z) = -\frac{1}{(2\pi)^2} \int_{R^2} \frac{e^{-i\zeta z} e^{-i\xi x}}{\xi^2 + m^2 \left(\zeta + wM/m^2 \right)^2 - (w/m)^2} d\xi d\zeta =$$

$$= -\frac{e^{-i(wMz/m^2)}}{(2\pi)^2 m} \int_{R^2} \frac{e^{-i\zeta z/m} e^{-i(\xi, x)}}{\xi^2 + \zeta^2 - (w/m)^2} d\xi d\zeta. \quad (37)$$

Variable substitution was also used here $\zeta = m(\zeta - wM/m^2)$. Here, under the sign of the integral is the Fourier transform of the fundamental solution of the two-dimensional Helmholtz equation:

$$\Delta \Phi + k^2 \Phi = \delta(y), \quad k = \frac{w}{m}, \quad y \in R^2,$$

The fundamental solution of this equation, satisfying the conditions of Sommerfeld radiation [16,17], taking into account the time factor, has the following form:

$$\Phi_2(y) = -\frac{i}{2\pi} H_0^{(2)}(k\|y\|), \quad y \in R^2;$$

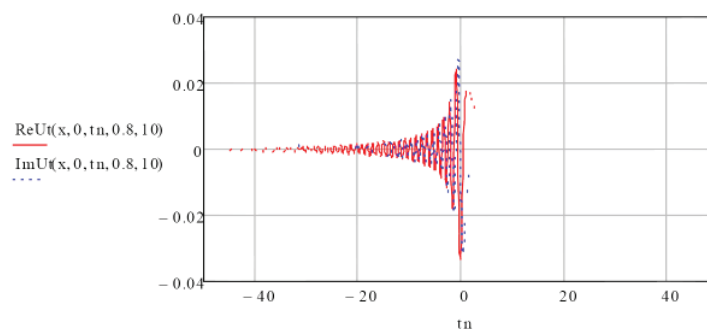


Figure 2: Temporal waveform of the vibrotransport signal at a fixed observation point along the X_3 -axis for a subsonic moving source (Mach number $M = 0.8$). The signal is plotted as a function of time $t = t_n$, showing the variation of wave amplitude and frequency as the source approaches and recedes from the observation point. The figure illustrates the Doppler effect, where both frequency and amplitude increase during approach and decrease during recession. The vibration frequency of the source is denoted by ω , and the propagation dynamics reflect the influence of source motion on the observed signal characteristics.

Here $H_0^{(2)}$ is the Hankel function of the second kind. Accordingly, comparing with the integral function in (37), taking into account linear transformations of variables, we obtain the original:

$$U(x, z) = -\frac{ie^{-iwm^{-2}Mz}}{(2\pi)^3 m} H_0^{(2)}\left(\frac{w}{m^2} \sqrt{z^2 + m^2 x^2}\right) \quad (38)$$

5.2. Solutions of homogeneous VTE at N=2

We now construct solutions of a homogeneous VTE:

$$\frac{\partial^2 u^0}{\partial x^2} + 2i w M \frac{\partial u^0}{\partial z} + w^2 u^0 = 0, \quad x \in R^1, z \in R^1; \quad (39)$$

In the space of Fourier transforms, it has the form:

$$-(\xi^2 + (1 - M^2)\zeta^2 - 2wv\zeta - w^2)\bar{u}^0 = 0 \quad \xi \in R^1, \zeta \in R^1 \quad (40)$$

The solution to this equation $\bar{u}^0 = \alpha(\xi, \zeta)\delta_S(\xi, \zeta)$ is a singular generalized function - a simple layer on the surface of an ellipsoid S , the center of which is shifted to a point $(0, 0, \zeta = wM / m^2)$:

$$\xi^2 + m^2 \left(\zeta - wM / m^2\right)^2 = (w / m)^2,$$

Here, the density of a simple layer $\beta(\xi, \zeta)$ is an arbitrary function integrable on S .

Accordingly

$$u^0(x, z) = \int_S \beta(\xi, \zeta) e^{-i(\xi, x)} e^{-i\zeta z} dS(\xi, \zeta) \quad (41)$$

Solutions of the homogeneous Helmholtz equation can also be used to construct $u^0(x, z)$:

$$\Delta u^0(y) + k^2 u^0(y) = 0, \quad y = (y_1, y_2)$$

They can be decomposed into Fourier-Bessel series:

$$u(y) = \sum_n b_n J_n(k \|y\|) e^{in\varphi}, \quad \|y\| = \sqrt{y_1^2 + y_2^2}$$

Since here $y = (x, z/m)$, we get

$$u^0(x, z) = e^{-i(wMz/m^2)} \sum_n b_n J_n\left(\frac{w}{cm} \sqrt{z^2 + m^2 r^2}\right) \frac{(x + iz / m)^n}{r^n}, \quad r = \sqrt{(z / m)^2 + x^2} \quad (42)$$

where the coefficients b_n are arbitrary complex numbers.

5.3. The general solution of the VTE at N = 2

Similarly to clause 4.3, the following theorem is proved.

Theorem 2. The solution of VTE (12) in 2D space has the following form:

$$u(x, z) = U(x, z) * g(x, z) + u^0(x, z). \quad (43)$$

If $g(x, z)$ is a regular function and $g(x, z) \in L_1(R^2)$, then

$$U(x, z) * g(x, z) = \int_{R^2} U(x - y, z - h) g(y, h) dy dh \quad (44)$$

If $g(x, z)$ - a singular function centered on the curve l : $g(x, z) = \beta(x, z)\delta_l(x, z)$, $\beta(x, z) \in L_1(l)$, then

$$U(x, z) * g(x, z) = \int_l U(x - y, z - h) g(y, h) dl(y, h) \quad (47)$$

If it $g(x, z) = G \frac{\partial^{n+m}}{\partial x^n \partial z^m} \delta(x, z)$ is a concentrated vibration transport source, then

$$U(x, z) * g(x, z) = G \frac{\partial^{n+m}}{\partial x^n \partial z^m} U(x, z).$$

Formula (43) makes it possible to determine the field of any vibration source from the class of generalized slow-growth functions, both regular and singular. At the same time, for singular functions, when calculating convolution, one should use the definition of convolution in the space of generalized functions [16,17].

6. One-dimensional VTE solutions for the motion of regular and singular vibration sources, N=1

6.1. Green's function and solutions of homogeneous VTE at N=1

In this case $u = u(z)e^{i\omega t}$, $\omega = \omega / c$, and $u(z)$ satisfies the equation

$$m^2 \frac{\partial^2 u}{\partial z^2} + 2i\omega M \frac{\partial u}{\partial z} + \omega^2 u = g(x, z), \quad z \in R^1; \quad (48)$$

The fundamental solution satisfies the equation:

$$m^2 \frac{\partial^2 U}{\partial z^2} + 2i\omega M \frac{\partial U}{\partial z} + \omega^2 U = \delta(z), \quad z \in R^1; \quad (49)$$

its Fourier transformant has the form:

$$\begin{aligned} \bar{U} &= -\frac{1}{m^2 \zeta^2 - 2\omega M \zeta - \omega^2} = -\frac{1}{m^2 \left(\zeta - \omega M / m^2\right)^2 - (\omega / m)^2} = \\ &= -\frac{m^{-2}}{\left(\zeta - \omega M / m^2\right)^2 - (\omega / m^2)^2}, \end{aligned} \quad (50)$$

To construct the original solution, we will use the fundamental solution of the ordinary differential equation (ODE):

$$\frac{d^2 \Phi_3}{dy^2} + \kappa^2 \Phi_3(y) = \delta(y), \quad \kappa = \omega / m^2;$$

$$\bar{\Phi}_3(\zeta) = -\frac{1}{\zeta^2 - \kappa^2}$$

the Green function of which has the form:

$$\Phi_3(y) = \frac{\sin(\kappa|y|)}{2\kappa}.$$

It does not tend to zero at infinity. But its amplitude decreases with an increase in the frequency of vibration, and *vice versa* increases with its decrease.

From this formula and formula (50), taking into account the shift property in the space of Fourier transforms, we obtain the original:

$$U(z) = \frac{\sin(\omega|z|/m^2)}{2\omega} e^{-i\omega M/m^2}.$$

Accordingly, the solution of a homogeneous VTY has the form:

$$u^0(x) = (a \cos(\kappa z) + b \sin(\kappa z)) e^{-iM/m^2}$$

6.2. The general solution of the VTY at $N = 1$

Similarly to paragraphs 4.3 and 4.5, the following theorem is proved.

Theorem 3. *The solution of VTE (12) in 2D space has the following form:*

$$u(z) = U(z) * g(z) + u^0(z)$$

If $g(z)$ is a regular function and $g(z) \in L_1(R^1)$, then

$$U(z) * g(x, z) = \int_{R^2} U(z - y) g(y) dy$$

If it $g(z) = G \frac{d^m \delta(z)}{dz^m}$ is a concentrated vibration transport source, then

Thus, all solutions of this equation in spaces of physical dimension are constructed. By analogy, they can be constructed in spaces of any dimension, which can be offered to an interested reader. Here we have limited ourselves to three.

Limitations of the study

The present study focuses mainly on the analytical construction of vibrotransport solutions within the framework of idealized hypotheses. The model considers a constant speed and frequency of the mobile source, which may not fully reflect the complexity of real systems, where these parameters may vary over time. In addition, the analysis is limited to homogeneous and isotropic media and therefore does not take into account heterogeneous or anisotropic properties likely to significantly influence wave propagation.

In addition, this work emphasizes the mathematical formulation and the construction of solutions, with limited consideration of experimental validation or numerical simulations. The physical interpretation of certain phenomena, such as the Doppler effect, remains presented in a simplified way and could require further study for specific applications. These limitations pave the way for future expansions to more general, application-oriented models.

Conclusion

The study of wave propagation processes in continuous media and electromagnetic fields leads to the solution of systems of partial differential equations of various types and the definition of their solutions in the form of vector fields that describe various characteristics of dynamic processes. These can be, for example, displacements and velocities, as in elastic and multicomponent media, or the intensity of electromagnetic fields, the change of which in space and time allows us to model such processes and study them using mathematical methods.

As is known, any vector field $\mathbf{u}(\mathbf{x}, t) = u_j(\mathbf{x}, t) \mathbf{e}_j$ can be represented through scalar and vector potentials (φ, Ψ) in the form [20]:

$$\mathbf{u}(\mathbf{x}, t) = \text{grad } \varphi(\mathbf{x}, t) + \text{rot } \psi(\mathbf{x}, t) \quad (51)$$

which describe dilation and vortex waves in the medium under consideration. In isotropic media, as a rule, they satisfy the wave equations:

$$\in_{c_\varphi} \varphi = f(\mathbf{x}, t), \quad \in_{c_\psi} \psi = \mathbf{g}(\mathbf{x}, t) \quad (52)$$

Since the speed of wave propagation in such media is always finite and does not depend on the direction of wave propagation. The velocity of motion can be different for these waves, as in elastic media, where shear waves described by a vector potential propagate slower than dilation waves. And in the electromagnetic environment described by Maxwell's equations, they are the same.

The vibration transport solutions of the wave equation constructed here make it possible to study wave processes in such media under the influence of mobile vibration sources of waves of various nature. In particular, solutions of the Lamé equations of elasticity theory using Lamé potentials, which satisfy (52), make it possible to study the stress-strain state of an elastic medium in such dynamic processes with wide application in problems of geophysics and seismology.

The constructed solutions can be used to solve vibration-transport boundary value problems of acoustics, elasticity theory and electrodynamics, as discussed in [5–7].

Note also that the constructed solutions at zero vibration frequency describe subsonic wave transport solutions, which were already well studied by the author earlier [1–7]. And transport and vibration loads are one of the most common sources of disturbances in environments. For example, the electromagnetic fields of electromagnetic emitters on mobile platforms, which are widely used in road and rail transport, can be modeled using the solutions built here.

Acknowledgement

This work was carried out with financial support from the Committee of Science of Ministry of Science and Higher Education of the Republic of Kazakhstan (grants: AP23488145, 2024–2026; AP19674789, 2023–2025).

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