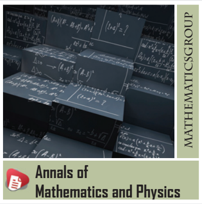

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Conceptual Review

Twice Punctured Euclidean and Hyperbolic Manifolds, Revisited as a Hypothetical Explanation for Quantum Dots

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(Presented first in Algebra and Geometry Seminar, 05 March, 2024, and at international conferences, including the XII BGL Conference in Budapest (1–3 May 2024), as a CERN Indico events)

A 40-Year Retrospective Honoring the 2023 Nobel Prize Laureates in Chemistry, and also the 200th anniversary of János Bolyai's work on absolute geometry

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Keywords: Manifold; Twice punctured (almost) manifold; Euclidean and hyperbolic crystal group; Quantum dots

Abstract

Upon reading a newspaper article from December 2023, I realized that my 1984 conference paper might provide a possible explanation relevant to the Nobel Prize in Chemistry awarded to Alexey Yekimov, Luis E. Brus, and Moungi G. Bawendi. After corresponding with my colleague, Academician István Hargittai, he extended his congratulations. I subsequently presented the topic at several conferences in 2024, including those held in Budapest, Sopron, Belgrade, and Senj.

Naturally, the author could not have anticipated the relevance and significance of the topic at that time. This was an incidental consequence of my earlier erroneous paper [2], intended to construct an infinite series of non-orientable compact hyperbolic manifolds as a polyhedral tiling series in the Bolyai-Lobachevsky hyperbolic space H3. Fortunately, I observed and improved the mistake soon. Specifically, those constructions were not manifolds because the two fixed-point orbits represent punctures where point reflections (central inversions) occur in the symmetry group of the tricky polyhedral tilings.

But these singular points, interpreted as 'quantum dots' (e.g., copper and chlorine ions), respectively, in a silicon-based glass fluid that subsequently solidifies produce optical effects (due to electron transitions) whose colors might depend on the sizes of crystal particles.

This suggests that the unexpected result was more significant than the original intention that could be reached easily later!

The main point of this paper is that – in addition to an Euclidean crystal group **61. Pbca** – I constructed infinitely many hyperbolic space groups (in Bolyai-Lobachevsky space) possibly providing such a material.

1. Introduction and original background documents (from [1])

A complete connected Riemannian n-dimensional manifold of constant sectional curvature is briefly called a space form. Intuitively, each space form is locally isometric to one of the classical n-spaces of constant curvature. It is well-known that each space form can be represented as an orbit space M/G.

Manifolds and 'Twice Punctured Manifolds'

Here M is one of simply connected n-spaces of curvature K, i.e. M is either a spherical ($K > 0$) or the Euclidean ($K = 0$) or a hyperbolic n-space ($K < 0$). The isometry group G acts discontinuously and freely on M, i.e. there is a nonempty open set V in M so that no two distinct points of V are equivalent under G, moreover, the identity 1 is the only element of G which has fixed points. Then G can be considered as the fundamental group of the manifold M/G.

There are 'twice punctured manifolds' which have two exceptional singular points with„half-ball-like" neighbourhoods, where the centrally opposite points are glued together. Such a fundamental group G has two singular orbits; that is, M/G is 'twice punctured'.

1.1. Dia 6. Euclidean example 61. Pbca

Orientation-preserving transformations:

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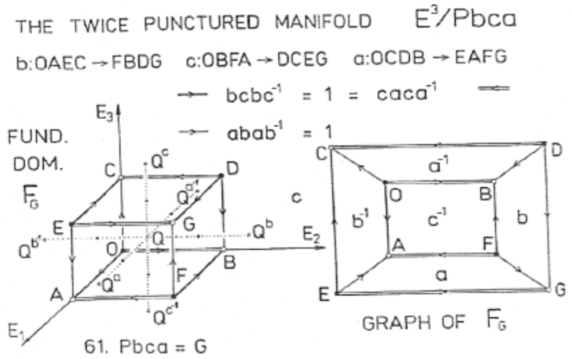
1 (x^1, x^2, x^3) the *identity* --- s_1 $(½ + x^1, ½ - x^2, -x^3)$; s_2 $(-x^1, ½ + x^2, ½ - x^3)$; s_3 $(½ - x^1, -x^2, ½ + x^3)$ represent screw motions

Orientation reversing transforms:

-1 $(-x^1, -x^2, -x^3)$ a point reflection (at the origin) --- b $(½ - x^1, ½ + x^2, x^3)$: *OAEC =:* $b^{-1} \rightarrow$ *FBDG =: b;* c $(x^1, ½ - x^2, ½ + x^3)$; a $(½ + x^1, x^2, ½ - x^3)$ *glide reflections*



A supergroup occurs: 205. **Pa3**$^{-}$ =: $G^{\sim}$, *If* F_G *is a cube, then a threefold rotation occurs around OG.*

Then at later presentations a movable "wonder cube" model (recently presented to me by Endre Budai, a teacher colleague in Teleki Blanka Gymnasium of Székesfehérvár, provided the following internet link https://www.thingiverse.com/thing:10483) incidentally illustrated this nano-size situation. The opposite vertices (with bigger part (Cu), smaller part (Cl) can be (trigonally) rotated.

Péter SZABÓ, University of Sopron, produced this cube model using a 3D printer in more variants.

1.2. To 61. Pbca = G in figure of former dia 6

The *fundamental domain (asymmetric unit)* F_G of 61. Pbca = G geometrically describes this group (e.g. in [4]) in the *orthorhombic* coordinate system *OE1E2E3*, where the *lengths* of *basis vectors* $OE_i := |e_i| = a_i$ $(i = 1, 2, 3)$ are given parameters (*by measuring the material crystal, to be determined*). An *orthorhombic lattice* ΛG of G is given by *integer coordinate triple* to the identity transform 1 (as linear part). as defined in our conventions, each α(A, a) ∈ G can be given by mapping $\propto: X \mapsto XA + a =: X\propto = Y$ with A by linear integer unimodular matrix to the basis (ei = OEi) above, and the translational component a, as illustrated in dia 6, the position vector is $OX = X = x^i e_i$ (Einstein-Schouten summing index convention), the above Y is the image.

Assume that O, D, E, F are G-equivalent point reflection centres, say with copper (Cu) ion parts, and similarly G, A, B, and C., are with Chlorine (Cl) ion parts, so that the fundamental cube F_G contain also "central" silicon (Si) atom in OG and 2-2 ones at the 3 opposite face pairs of F_G, equivalent by glide reflections b, c, a, respectively. Assume that these (approximate) proportions 1/2 : 1/2 : 7(?) of atoms Cu, Cl, Si can form crystal particles of appropriate cubic size, and these particles float in a silicon fluid that freezes. The singularities (as constraints) near Cu and Cl ions cause electrons "jumping-leaping" with light effects, i.e. quantum dots. This may have applications in display technologies. e.g. TV screen.

2. Hyperbolic series

2.1. Fundamental domain series F^1_{tu} of groups G^1_{tu} in hyperbolic space H^3

Generators: a_i $(-t \le i \le +t)$: $a_i^{-1} \rightarrow a_i$ *glide reflections* --- p_i $(-t \le i \le +t)$: $p_i^{-1} \rightarrow p_i$ *screw motions* --- r_i $(-t \le i \le +t)$: $r_i^{-1} \rightarrow r_i$ screw motions ---

s_u $(0 \le u \le t$ *with fixed u*$)$: $s_u^{-1} \rightarrow s_u$ *a unique screw motion --- Altogether, there are* $3q + 1 = 6t + 4$ *generators.*

Relations are represented by arrowed edges in Table 1:

e.g. $\Rightarrow a_{-t}a_{-t}\ldots a_0a_0\ldots a_{+t}a_{+t} = \mathbf{1}$ identity; --- e.g. ooo> $a_0p_0a_{-u}^{-1}r_t = \mathbf{1}$

The equivalence classes C (**Cu**) and D (**Cl**) are *reflections centres with half ball neighbourhoods*, i.e., *punctures for quantum dots.*

The other point classes (orbits, e.g. for G, E, A, H, L, ... for occasional **Si** atoms (ions)) have ball neighbourhoods, as at manifolds.

The *proportions* depend on parameter q = 2t + 1. We have infinitely many hyperbolic possibilities. The minimal one with t = 1, *q* = 3 is

potentially significant for experimental realization. For instance, it follows in case q = 3, u = 0 on the base of Table 1:

$$\mathbf{G^1_{10} = \{a_{-1}, a_0, a_1; s_0 =: s, p_{-1}, p_0, p_1, r_{-1}, r_0, r_1 \text{ -- } 1 = a_{-1}^2a_0^2a_1^2 = (a_{-1}sa_1s^{-1})(a_0sa_{-1}s^{-1})(a_1sa_0s^{-1})}$$

$$\mathbf{= (p_{-1}sr_{-1}^{-1}s) = (p_0sr_0^{-1}s) = (p_1sr_1^{-1}s)}$$

$$\mathbf{= (a_{-1}p_{-1}a_{-1}^{-1}r_0) = (a_0p_0a_0^{-1}r_1) = (a_1p_1a_1^{-1}r_{-1})}$$

$$\mathbf{= (a_{-1}r_1^{-1}a_{-1}^{-1}p_{-1}^{-1}) = (a_0r_{-1}^{-1}a_0^{-1}p_0^{-1}) = (a_1r_0^{-1}a_1^{-1}p_1^{-1})\}}$$

with trigonal rotational symmetric fundamental domain.

Remark. It was a tedious work, with lengthy computations

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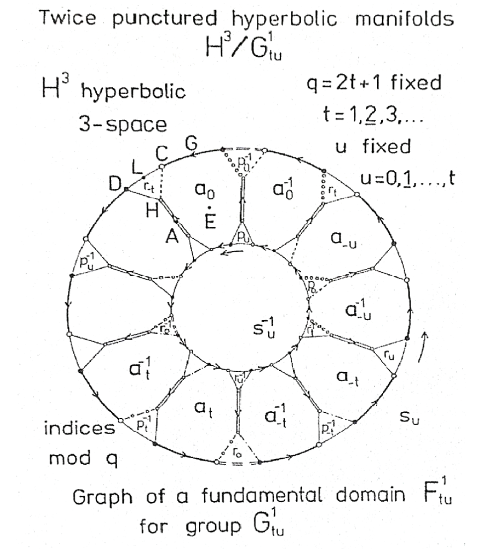
in our original paper [1] that both points C and D have fixing point reflections. Extra group computation was, how to derive this from the presentation of $\mathbf{G}^1_{10}$ in Table 1 (This can be verified using the above example).



2.2. Fundamental domain series F^2_t of G^2_t (u = 0) in H^3

Generators: $\mathbf{a}_i(-t \le i \le +t)$: $a_i^{-1} \to a_i$ *glide reflections* --- $\mathbf{c}_i$ $(-t \le i \le +t)$: $c_i^{-1} \to c_i$ *glide reflections* --- $\mathbf{d}_i$ $(-t \le i \le +t)$: $d_i^{-1} \to d_i$ *glide reflections*---

$\mathbf{e}$ (0 ≤ u ≤ t, *for simplicity, u = 0 is fixed to obtain a unique case) glide reflection* --- Altogether: 3q + 1 = 6t + 4 generators.

Relations are collected to arrowed edges in Table 1:

e.g. ⇒ $\mathbf{a}_{-t}\mathbf{a}_{-t}\ldots\mathbf{a}_0\mathbf{a}_0\ldots\mathbf{a}_{+t}\mathbf{a}_{+t} = \mathbf{1}$ identity; --- e.g. ooo> $\mathbf{a}_0\mathbf{c}_0\mathbf{a}_0^{-1}\mathbf{d}_t = \mathbf{1}.$

The equivalence classes C (**Cu**) and D (**Cl**) are reflection centres with *half ball neighbourhoods, i.e., punctures for quantum dots.*

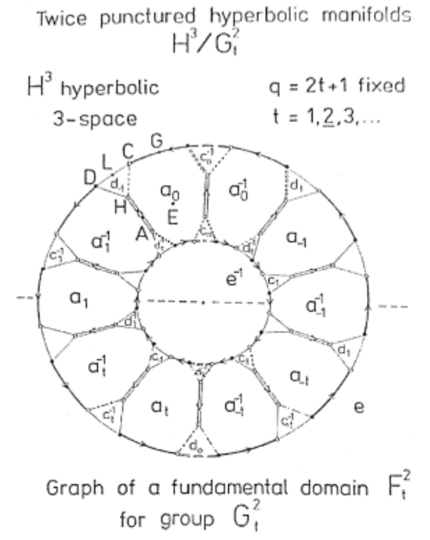
An integer parameter u with 0 ≤ u ≤ t can be introduced for other (non-isometric) manifolds, similarly as before.

The (projective) metric, i.e. Beltrami-Cayley-Klein (B-C-K) model, is introduced later for Bolyai - Lobachevsky hyperbolic geometry!

Then we can compute all angle and distance data of the above and following (congruent for equal t-s) fundamental domains.

2.3. Reflection group C_t and its fundamental domain in H^3 as truncated orthoscheme; hints to B-C-K model of H^3, generators for previous G^1_{tu} and G^2_t; some distances for *Coxeter* diagram, sketch.

The orthoscheme projective coordinate simplex $A_0A_1A_2A_3$ ~ $b^0b^1b^2b^3$ *will be described in the real (left) vector 4-space* $\mathbf{V}^4$ (for points X (where X = X^iA_i ~ cX) and its dual (right) form space V_4 (for planes *u*(**u** = b^ju_j ~ *uc*); $A_ib^j = \delta_i^j$ *(Kronecker symbol).*



By the *symmetric Coxeter– Schläfli* (C–Sch) *matrix* $(b^{ij}) = \langle b^i, b^j\rangle = (\cos(\pi - \beta^{ij}))$

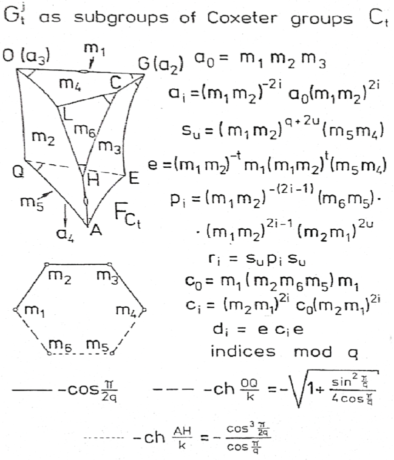
first for angles $\beta^{ij} = (\angle b^ib^j)$ with $(\angle b^ib^i) = \pi$, as is standard; then for distances by the inverse $(b^{ij})^{-1} =: (A_{ij}) =: \langle \mathbf{A}_i, \mathbf{A}_j\rangle$, so

$$\cosh(XY/k) = -\langle \mathbf{X}, \mathbf{Y}\rangle/(\langle \mathbf{X}, \mathbf{X}\rangle\langle \mathbf{Y}, \mathbf{Y}\rangle)^{1/2}$$

is the distance of points X and Y. Here $k = (-1/K)^{1/2}$ is the universal unit distance of the hyperbolic space $\mathbf{H}^3$, K is the constant negative sectional curvature.

In nano size k must be determined. The Periodic Table of the Elements gives important information by relative atomic weights, then consequently atomic radii! Further mathematical details are provided in the References, e.g. [1-9].

In our next Figure (from the original paper [1]) the doubly truncated orthoscheme comes from the $A_0A_1A_2A_3 = b^0b^1b^2b^3$ coordinate simplex (see also later Figure 3, 1), but here we used the previous notations of paper [1]: $\mathbf{A}_2 \to \mathbf{a}_3$ for point O, $\mathbf{A}_1 \to \mathbf{a}_2$ for point G, $\mathbf{A}_3 \to \mathbf{a}_4$ for outer point A_3, whose polar plane a_3 is m_5 here, A_0 is outer point whose polar plane a_0 is denoted by m_6 here. The simplex planes b^0, b^1, b^2, b^3 are denoted by m_1, m_2, m_3, m_4 now.



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2.4. Table 1: Presentations of fundamental groups G^1_{tu} and G^2_t by generators and defining relations to previous figures.

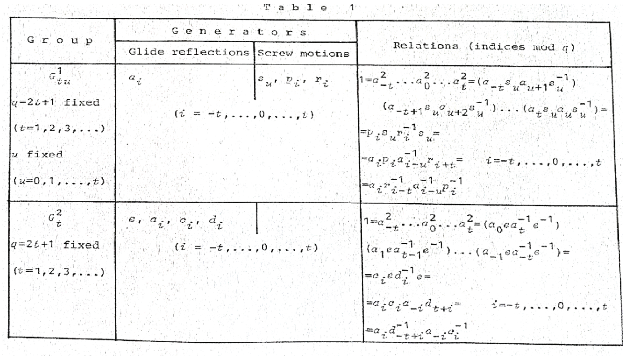


Table 1

Group	Generators: Glide reflections	Generators: Screw motions	Relations (indices mod q)
G^1_{tu} $q=2t+1$ fixed $(t=1,2,3,\dots)$ u fixed $(u=0,1,\dots,t)$	a_i $(i=-t,\dots,0,\dots,t)$	s_u, p_i, r_i	$1=a^2_{-t}\dots a^2_0\dots a^2_t=(a_{-t}s_ua_{u+1}s_u^{-1})$ $(a_{-t+1}s_ua_{u+2}s_u^{-1})\dots(a_ts_ua_us_u^{-1})=$ $=p_is_ur_i^{-1}s_u=$ $=a_ip_ia^{-1}_{i-u}r_{i+t}=$ $\quad i=-t,\dots,0,\dots,t$ $=a_ir^{-1}_{i-t}a^{-1}_{i-u}p_i^{-1}$
G^2_t $q=2t+1$ fixed $(t=1,2,3,\dots)$	e, a_i, c_i, d_i $(i=-t,\dots,0,\dots,t)$		$1=a^2_{-t}\dots a^2_0\dots a^2_t=(a_0ea_t^{-1}e^{-1})$ $(a_1ea^{-1}_{t-1}e^{-1})\dots(a_{-1}ea^{-1}_{-t}e^{-1})=$ $=c_ied_i^{-1}e=$ $=a_ic_ia_{-i}d_{t+i}=$ $\quad i=-t,\dots,0,\dots,t$ $=a_id^{-1}_{-t+i}a_{-i}c_i^{-1}$

3. The Euclidean cube tiling and its characteristic orthoscheme (4, 3, 4), C-Sch diagram, matrix (illustration for later analogous hyperbolic projective metric)

The matrix (b^{ij}) in the next figure is of signature $(+, +, +, 0)$ indeed for euclidicity. Then the signature depends on parameters u, v, $w = u$, as crucial. For hyperbolicity $(+, +, +, -)$ is necessary later on [10,11].

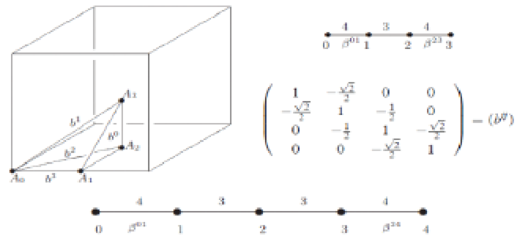


Figure 3: Cube tiling in $\mathbf{E}^3$ and symbols for it. Coxeter-Schläfli diagram for the $\mathbf{E}^4$ cube tiling.

3.1. C-Sch matrix and its inverse for orthoscheme (u, v, w(=u)) and for "trunc-simplices". scalar products for forms (planes) and vectors (points)

$$(b^{ij}) = \langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle := \begin{pmatrix} 1 & -\cos\frac{\pi}{u} & 0 & 0 \\ -\cos\frac{\pi}{u} & 1 & -\cos\frac{\pi}{v} & 0 \\ 0 & -\cos\frac{\pi}{v} & 1 & -\cos\frac{\pi}{w} \\ 0 & 0 & -\cos\frac{\pi}{w} & 1 \end{pmatrix}.$$

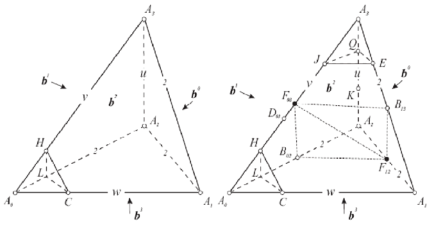


Figure 1: Simple and double truncated complete orthoschemes

$$(a_{ij}) = (b^{ij})^{-1} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle :=$$
$$= \frac{1}{B}\begin{pmatrix} \sin^2\frac{\pi}{w}-\cos^2\frac{\pi}{v} & \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} \\ \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \sin^2\frac{\pi}{w} & \cos\frac{\pi}{v} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{v} & \sin^2\frac{\pi}{u} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} & \sin^2\frac{\pi}{u}-\cos^2\frac{\pi}{v} \end{pmatrix}, \tag{2.2}$$

where

$$B = \det(b^{ij}) = \sin^2\frac{\pi}{u}\sin^2\frac{\pi}{w}-\cos^2\frac{\pi}{v} < 0 \ \text{ or } \ \sin\frac{\pi}{u}\sin\frac{\pi}{w}-\cos\frac{\pi}{v} < 0.$$

3.2. The volume of the orthoscheme by N. I. Lobachevsky's ideas with generalisation of R. Kellerhals

Theorem 2.2 (R. Kellerhals) *The volume of a three-dimensional hyperbolic complete orthoscheme* $\mathcal{O} - W_{uvw} \subset \mathbf{H}^3$ *is expressed with the essential angles* $\alpha_{01} = \frac{\pi}{u}$, $\alpha_{12} = \frac{\pi}{v}$, $\alpha_{23} = \frac{\pi}{w}$, $(0 \le \alpha_{ij} \le \frac{\pi}{2})$ *(Fig. 1.a, b) in the following form:*

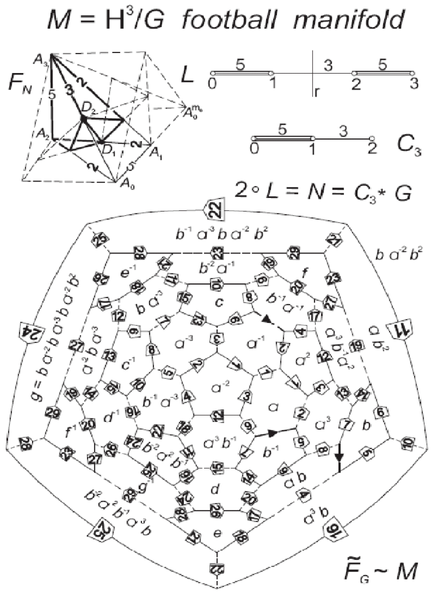
$$\mathrm{Vol}(\mathcal{O}) = \frac{1}{4}\{\mathcal{L}(\alpha_{01}+\theta)-\mathcal{L}(\alpha_{01}-\theta)+\mathcal{L}(\frac{\pi}{2}+\alpha_{12}-\theta)+$$
$$+\mathcal{L}(\frac{\pi}{2}-\alpha_{12}-\theta)+\mathcal{L}(\alpha_{23}+\theta)-\mathcal{L}(\alpha_{23}-\theta)+2\mathcal{L}(\frac{\pi}{2}-\theta)\},$$

where $\theta \in [0, \frac{\pi}{2})$ *is defined by:*

$$\tan(\theta) = \frac{\sqrt{\cos^2\alpha_{12}-\sin^2\alpha_{01}\sin^2\alpha_{23}}}{\cos\alpha_{01}\cos\alpha_{23}},$$

and where $\mathcal{L}(x) := -\int_0^x \log|2\sin t|dt$ *denotes the Lobachevsky function.*

3.3. The football polyhedron {5, 6, 6} from the half orthoscheme (5, 3, 5)



For "hyperbolic football" u = w = 5, v = 3, the signature of the C-Sch matrix is $(+, +, +, -)$, indeed. The 3 -▶- arrows determine the face pairing generators **a**: $a^{-1} \rightarrow a$ and **b**: $b^{-1} \rightarrow b$ and the product **ab**.

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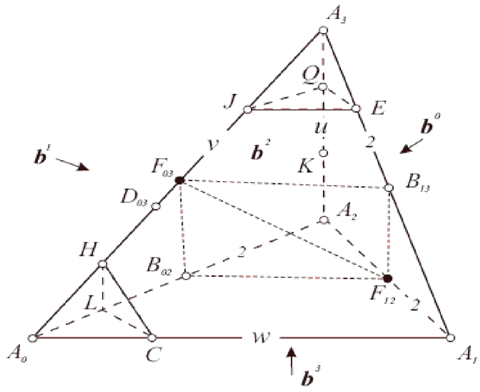
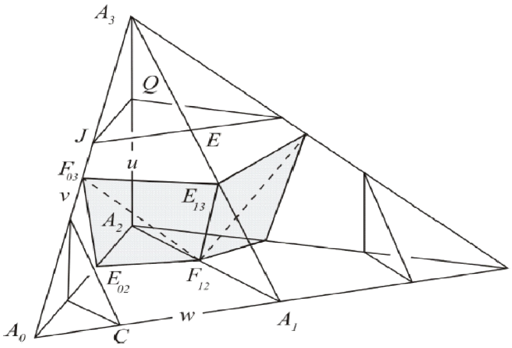
E. Molnár, Two hyperbolic football manifolds. In: Proceedings of *International Conference on Differential Geometry and Its Applications*, Dubrovnik Yugoslavia, 1988. 217–241.

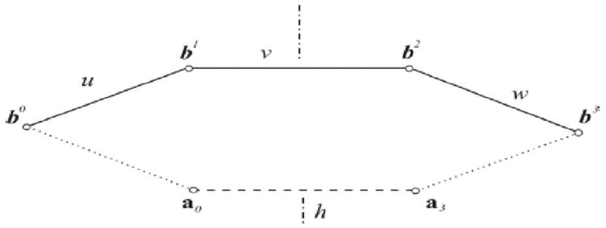
E. Molnár, On non-Euclidean crystallography, some football manifolds, *Struct Chem* (2012) 23:1057–1069.

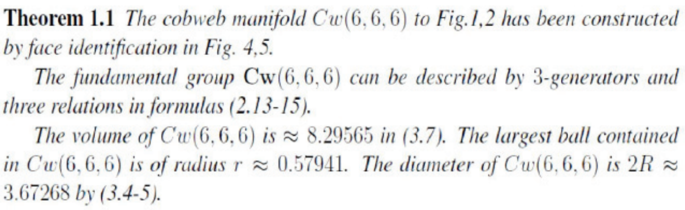
4. Construction of orientable cobweb (or tube) manifold series for visualisation from half trunc-orthoscheme, by 2z = u = v = w ≧ 6

Trunc-simplex (u, v, $w = u$) with (1/u) + (1/v) < ½; then A_0 and A_3 are outer vertices with truncating polar planes a_0 resp. a_3







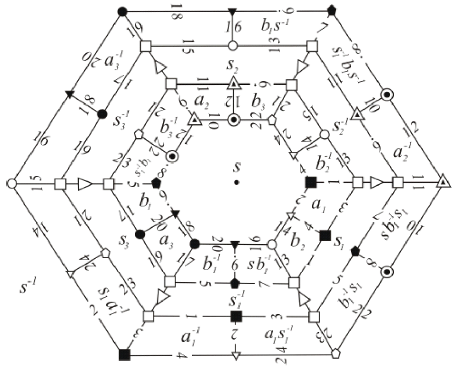


Theorem 1.1 *The cobweb manifold* $Cw(6,6,6)$ *to Fig.1,2 has been constructed by face identification in Fig. 4,5.*

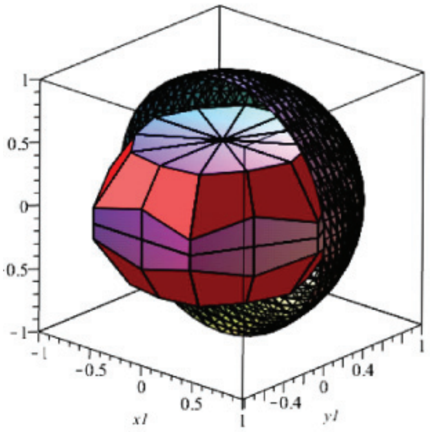
The fundamental group $\mathbf{Cw}(6,6,6)$ *can be described by 3-generators and three relations in formulas (2.13-15).*

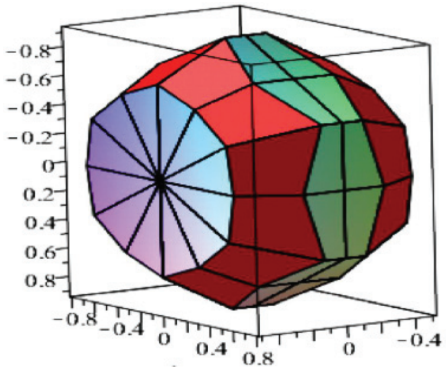
The volume of $Cw(6,6,6)$ *is* ≈ 8.29565 *in (3.7). The largest ball contained in* $Cw(6,6,6)$ *is of radius* $r \approx 0.57941$*. The diameter of* $Cw(6,6,6)$ *is* $2R \approx 3.67268$ *by (3.4-5).*

4.1. Construction of cobweb (tube) manifold Cw(6) by complex face-pairing identifications from D-V cell of previous point Q



4.2. Images from the animation of cobweb (tube) manifolds Cw(6) in B-C-K model of H^3





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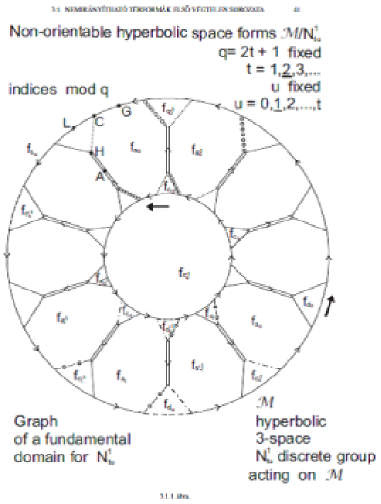
Appendix

New infinite series of non-orientable hyperbolic space forms as the original intention

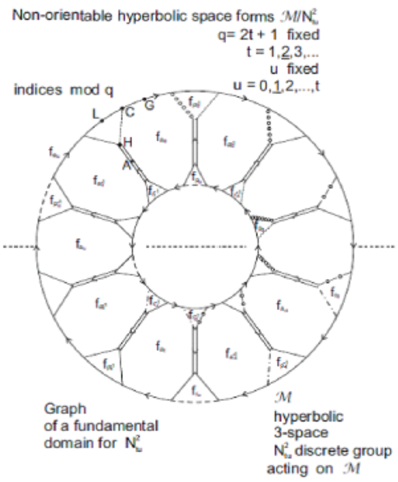
This section explains how the earlier issue was resolved, and obtain from *twice punctured manifold series a non-orientable manifold series* by modified face identifications. Indeed, only orbits C and D need to be merged, to get a ball-like neighbourhood for C = D orbit.

Changing the face identifications: screw motions ***p, r* to** glide reflections ***c, d (respectively)*** *in* G^1_{tu}, so C = D is glued together, we obtain non-orientable space form [3].



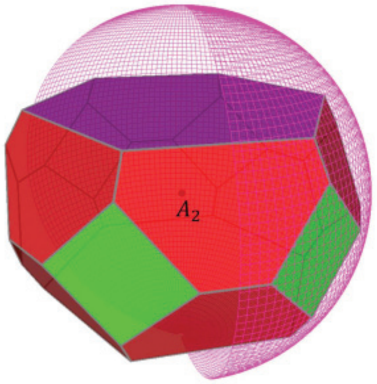
M /N^1_{tu} (M = H^3). Compare with the former **H^3/G^1_{tu}** (Here C = D)

Similarly, changing glide reflections **c, d** to screw motions **p, r** in **G^2_t**, so C = D is glued together, we obtain non-orientable space form *M /N^2_t* (Here M = **H^3**).

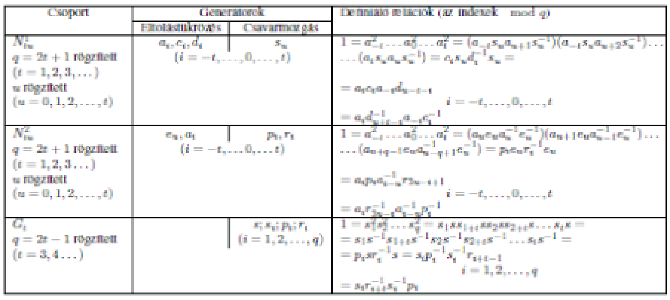


Compare with the former **H^3/G_t^2** (Here C = D)

Table 2: The algorithmic fundamental group presentations of our non-orientable manifold series and of their orientable twofold covering groups **G_t** (in Hungarian from [3], but as before in Table 1).



Another fundamental domain with trigonal rotation symmetry with other presentations (the figure constructed as a fine D-V cell by our Indonesian doctor student (now PhD) Arnasli Yahya)



Csoport	Generátorok		Definiáló relációk (az indexek mod q)
	Eltolástükrözés	Csavarmozgás	
N^1_{tu} $q=2t+1$ rögzített $(t=1,2,3,\dots)$ u rögzített $(u=0,1,2,\dots,t)$	a_i, c_i, d_i $(i=-t,\dots,0,\dots,t)$	s_u	$1=a^2_{-t}\dots a^2_0\dots a^2_t=(a_{-t}s_ua_{u+1}s_u^{-1})(a_{-t}s_ua_{u+2}s_u^{-1})\dots$ $\dots(a_ts_ua_us_u^{-1})=c_is_ud_i^{-1}s_u=$ $=a_ic_ia_{-i}d_{u-i-1}$ $i=-t,\dots,0,\dots,t$ $=a_id^{-1}_{u+i-t}a_{-i}c_i^{-1}$
N^2_{tu} $q=2t+1$ rögzített $(t=1,2,3\dots)$ u rögzített $(u=0,1,2,\dots,t)$	c_u, a_i $(i=-t,\dots,0,\dots,t)$	p_i, r_i	$1=a^2_{-t}\dots a^2_0\dots a^2_t=(a_uc_ua_u^{-1}c_u^{-1})(a_{u+1}c_ua^{-1}_{u-1}c_u^{-1})\dots$ $\dots(a_{u+q-1}c_ua^{-1}_{u-q+1}c_u^{-1})=p_ic_ur_i^{-1}c_u$ $=a_ip_ia^{-1}_{i-u}r_{2u-i+1}$ $i=-t,\dots,0,\dots,t$ $=a_ir^{-1}_{2u-i}a^{-1}_{i-u}p_i^{-1}$
G_t $q=2t-1$ rögzített $(t=3,4\dots)$		$s; s_i; p_i; r_i$ $(i=1,2,\dots,q)$	$1=s_1^2s_2^2\dots s_q^2=s_1ss_{1+t}ss_2ss_{2+t}s\dots s_qs=$ $=s_1s^{-1}s_{1+t}s^{-1}s_2s^{-1}s_{2+t}s^{-1}\dots s_qs^{-1}=$ $=p_isr_i^{-1}s=s_ip_i^{-1}s_i^{-1}r_{i+t-1}$ $i=1,2,\dots,q$ $=s_ir^{-1}_{i+t}s_i^{-1}p_i$

Our analogous non-orientable manifold fundamental group series **N^1_{tu}** and **N^2_{tu}** and their orientable twofold covering group series **G_t**

Of course, the reader needs more detailed explanations. Thus we (with Jenő Szirmai and Arnasli Yahya) turn back to the topic in a revisited mathematical paper.

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