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Review Article

On the Electron Spin Origin and Related Symmetry Aspects

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Abstract

We present a novel discussion of the electron spin origin, its symmetry properties and related conservation laws from a mathematical physics point of view, having put into the background the algebraic description of the corresponding physically observed representations. There is analyzed in detail, the spin structure and its crucial dependence on the $SU(2)-$ symmetry properties of the related representations of the basic Clifford algebra, generated by creation-annihilation operators on the Fock space and the related chirality symmetry of the Pauli spin operators. Based on the conservation law of the spin projection on the electron momentum, a novel derivation of the Dirac Hamiltonian operator, whose Lorentz invariance is naturally related to that of the fundamental Maxwell equations, whose quanta are carriers of interaction between electrons.

1. Introductory setting

This Letter is devoted to a novel discussion of the electron spin, spin conservation laws and derivation of the Dirac Hamiltonian operator from a mathematical physics point of view, having put into the background of our elementary enough analysis main electron symmetry properties, their algebraic description and the corresponding physically observed representations. Subject to our physical real spacetime under regard, we assume as a fact the existence of such an elementary particle as an electron, specified by its nonzero discrete charge parameter $\xi \in \mathbb{R}$, localized in the Minkowski [1,2] spacetime $M^4 = \mathbb{R}t \times \mathbb{R}^3_x$, where $x \in \mathbb{R}^3$ is the spatial variable, $t \in \mathbb{R}$ is the temporal variable. These coordinates are, obviously, important for specifying an electron by means of external and independent observables allowed by the physical world. We naturally accept that there also exist other elementary particles like protons, photons, mesons, etc., considered as quanta of the physical world structure, and which are indistinguishable, that is, identical within each particle kind. Moreover, we accept such experimentally approved important physical phenomena as the creation and annihilation of elementary quantum particles during their interaction in spacetime. At the beginning of the past 20 century both qualitative and quantitative properties of these quantum physical phenomena have been effectively enough described by classical theoretical physicists W. Pauli, E. Schrödinger, W. Heisenberg, P. Dirac [2-5]and others [6-10], within first quantum mechanics and next quantum field theory by means of modern mathematical tools, based on special differential-operator expressions, called "Hamiltonians", jointly with the related spectral theory of special linear differential-operator Schredinger equations [5,11-14] in the Hilbert space

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H of complex-valued functions $f:\mathbb{R}^3\to\mathrm{C}$, integrable with respect to the usual Lebesgue measure dx for $x\in\mathbb{R}^3$, and whose solutions possess almost all information about the evolution of quantum particle states in spacetime and their interaction with neighborhood. In particular, theoretical physicists soon enough understood that each quantum event in a microuniverse, related to such elementary particles, can be well modelled by means of some special operators on the Fock space consisting of the corresponding quantum states, and in the case of electrons, the most fundamental facts - their creations and annihilations in vacuum. These fundamental operator objects on the Fock space and the related symmetry properties of their algebraic representations proved to be governing in deepening our understanding of the electron spin nature and its importance for the matter stability in the Universe.

The elementary point charged particle, like an electron, its spin, energy spectrum and mass problem were inspiring many physicists from the past, such as J. J. Thompson, G.G. Stokes, H.A. Lorentz, E. Mach, M. Abraham, P.A. M. Dirac, G.A. Schott and their followers [15-20]. Nonetheless, their studies have not given rise up to date to a clear explanation of this phenomenon that stimulated researchers [21-26] to tackle it from different approaches based on new ideas stemming both from the classical Maxwell-Lorentz electromagnetic theory, as in [27-31], and modern quantum field theories [32-35] of Yang-Mills and Higgs type, as in [36-39] and others, whose recent and extensive review is done in [40].

In the work, I also concentrate on the electron spin structure and its deep connection with the symmetry properties of the related representations of the basic Clifford algebra, generated by creation-annihilation operators on the Fock space. Based on the special Clifford algebra representation corresponding to the Pauli $su(2)$ symmetry algebra generators, their chirality symmetry and the related temporal conservation of the spin projection on the electron momentum, a quantum Hamiltonian operator on the Fock space, whose finite-dimensional invariant projection coincides exactly with the classical Dirac operator, whose Lorentz invariance follows as a natural consequence.

2. Electron spin representation and related algebraic structures

Still, in 1932, the following, in some sense, virtual linear operators on a suitably chosen "*physical*" Hilbert space Φ of quantum electron states:

1) **creation operator**

$$\psi^+(x):\Phi\to\Phi, x\in\mathbb{R}^3 \tag{2.1}$$

of quantum states, assigned to an electron, localized in point $x\in\mathbb{R}^3$ and respectively,

2) annihilation operator

$$\psi(x):\Phi\to\Phi,\ x\in\mathbb{R}^3, \tag{2.2}$$

of quantum states, assigned to an electron, localized in point $x\in\mathbb{R}^3$ as the corresponding operator conjugation to the introduced above creation operator:

$$(\psi^+(x)h\,|\,g)_\Phi=(h\,|\,\psi(x))g)_\Phi \tag{2.3}$$

for all $x\in\mathbb{R}^3$ with respect to the scalar product $(\cdot\,|\,\cdot)_\Phi$ on Φ for all $|h)$, $|g)\in\Phi$ for which there exists a so-called "*vacuum*" vector $|\Omega_0)\in\Phi$, satisfying the determining relationship:

$$\psi(x)\,|\,\Omega_0)=0 \tag{2.4}$$

for all $x\in\mathbb{R}^3$.

It is worth noting that the annihilation operator simulates the real physical process of the annihilation of an electron at spatial point $x\in\mathbb{R}^3$, sending it into the vacuum, where countless electrons exist being unobserved Thus, one can create new quantum states $|\,\Omega_n)\in\mathcal{H}^{\otimes n}:=\underbrace{\mathcal{H}\otimes\mathcal{H}\otimes\cdots\otimes\mathcal{H}}_{n\ times}$ for $n\in\mathrm{N}$ electrons, located, respectively, at points $x_j\in\mathbb{R}^3, j=\overline{1,n}$, via acting on the vacuum state $|\Omega_0)$ of the Hilbert space Φ by means of the creation operators:

$$|\,\Omega_n(x_1,x_2,...,x_n):=\psi^+(x_n)\psi^+(x_{n-1})...\psi^+(x_1)\,|\,\Omega_0). \tag{2.5}$$

As a number $n\in\mathrm{N}$ of electrons is not fixed, all such quantum states (2.5) should be combined as the direct sum into the unique Hilbert space $\Phi\simeq\oplus_{n\in\mathbb{Z}_+}\mathcal{H}^{\otimes n}$, which was done for the first time by V. Fock [41] in 1932 and is called the *Fock space* of multiparticle quantum states. Really, if we introduce in the space Φ the scalar product

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$$(f\,|\,g)_{\Phi} := \Sigma_{n\in\mathbb{Z}_+}\frac{1}{n!}(f_n\,|\,g_n)_{\mathcal{H}^n} \qquad (2.6)$$

for any $f, g \in \Phi$, then any quantum state $f \in \Phi$ can be uniquely represented as

$$f = \Sigma_{n\in\mathbb{Z}_+}\frac{1}{\sqrt{n!}}\int_{\mathbb{R}^{3\times n}} f_n(x_1,x_2,...,x_n)\,|\,\Omega_n(x_1,x_2,...,x_n)\,_{j=\overline{1,n}}dx_j \qquad (2.7)$$

with coefficients $f_n\in\mathcal{H}^{\otimes n}, n\in\mathbb{N}$, where the basic quantum electron states $|\Omega_n)\in\Phi, n\in\mathbb{N}$, defined by the expression (2.5), can be taken to be orthonormalized and dense in $\mathcal{H}^{\otimes n}$.

Yet, now an interesting question arises: what is a nature of these multiparticle Fock subspaces $\Phi_n\simeq\mathcal{H}^{\otimes n}, n\in\mathbb{N}$, within which there can be created and annihilated many of non distinguished electrons at different points of space? Having restoring to the experimental fact [2,3,42] explained by W. Pauli [4] that the quantum n- electron states $|\Omega_n) \in \Phi_n$, $n \in$ N persist to be invariant under spatial replacement of the electrons up to the sign, meaning nothing else than these quantum states are skew-symmetric functions of their spatial arguments with respect to the usual permutation symmetry group Σ_n which acts on Φ_n the following way:

$$\sigma : |\,\Omega_n(x_1,x_2,...,x_n) \to (-1)^{\sigma}\,|\,\Omega_n(x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},...,x_{\sigma^{-1}(n)}) \qquad (2.8)$$

for any permutation $\sigma \in \Sigma_n$ with parity $(-1)^{\sigma}$ and $(x_1,x_2,...,x_n)\in\mathbb{R}^{3\times n}$.

Consider now the action of the creation operator $\psi^+(x):\Phi\to\Phi, x\in\mathbb{R}^3$, based on the definition (2.5):

$$\psi^+(x)\,|\,\Omega_n(x_1,x_2,...,x_n)) := |\,\Omega_{n+1}(x,x_1,x_2,...,x_n), \qquad (2.9)$$

One easily obtains that the related action

$$\Psi(x)\,|\,\Omega_n(x_1,x_2,...,x_n) = \Sigma_{j=\overline{1,n}}(\varepsilon)^{j-1}\delta(x-x_j)\,|\,\Omega_{n-1}(x_1,x_2,...,\hat{x}_j,...,x_n), \qquad (2.10)$$

Where ε = 1 for symmetric and ε = −1 for anti-symmetric Hilbert space $\mathcal{H}^{\otimes n}, n\in\mathbb{N}$. Based on the relationships (2.9) and (2.10), it is easy to check that the following algebraic commutation expressions

$$\psi(f_1),\psi(f_2)]_{\varepsilon} = 0 = [\psi^+(f_1),\psi^+(f_2)]_{\varepsilon}, \qquad (2.11)$$

$$\psi(f_1),\psi^+(f_2)]_{\varepsilon} = (f_1\,|\,f_2)_{\mathcal{H}}$$

hold for arbitrary $f_1, f_2 \in S(\mathbb{R}^3; \mathbb{C})$, where $[\cdot,\cdot]_{\varepsilon}$ denotes commutator for ε = 1 and anti-commutator for ε == −1, and for an operator $a(x):\Phi\to\Phi,\ \ x\in\mathbb{R}^3$, the notation $a(f):=\int_{\mathbb{R}^3} f(x)a(x)dx, f\in S(\mathbb{R}^3;\mathbb{C})$, means its smearing over $\mathbb{R}^3$ with the "density" $f\in S(\mathbb{R}^3;\mathbb{C})$. In addition, one easily states that the following operator

$$N := \int_{\mathbb{R}^3}\psi^+(x)\psi(x)dx \qquad (2.12)$$

on Φ is self-adjoint and integer-valued, counting amounts of particles, described by quantum states in Φ Since for such an elementary quantum particle as an electron, the case ε = 1 is realised that is, the electron is a particle, satisfying the Fermi type statistics, and the basic Hilbert subspaces $\mathcal{H}^{\otimes n}, n\in\mathbb{N}$, are skew-symmetric, on which there is realized a two-valued representation of the symmetric group Σ_n. The corresponding two-valued representation Fock space Φ^{el} can be, obviously, generated by the extended vacuum vector $|\,\Omega_0^{el}) := |\,\Omega_0)\otimes|\,\Omega_{int,0})\in\Phi^{el}\simeq\Phi\otimes\Phi_{int}$, which is isometrically equivalent to the extended Hilbert space $\Phi_{int}\simeq\mathbb{E}^2, \mathbb{E}^2 := (\mathbb{C}^2;\langle\cdot|\cdot\rangle)$, where I took into account that the corresponding internal basic Fock space $\Phi_{int}\simeq\mathbb{E}^2$.

To understand in more detail the structure of this Fock space regarding the real electron particle dynamics in the Minkowski [1] spacetime $M^4=\mathbb{R}_t\times\mathbb{R}_x^3$, I need to recall that its evolution is governed by means of the classical Schrödinger equation, characterized by the quantum Hamiltonian operator $H_f:\tilde{\Phi}^{el}\to\tilde{\Phi}^{el}$, whose eigenvalues correspond to the observed electron energy states. The latter, in particular, means that for arbitrary $n \in$ N the projection $H_f\,|_{\mathcal{H}^{\otimes n}} := H_{f,n}:\mathcal{H}^{\otimes n}\otimes\mathbb{E}^2\to\mathcal{H}^{\otimes n}\otimes\mathbb{E}^2$ of this Hamiltonian operator should be represented as an operator element of the product $Cl_{2n}(\psi,\psi^+)\otimes Cl_3(\sigma)$, where $Cl_{2n}(\psi,\psi^+)$ is the Clifford algebra, generated by the creation-annihilation operators $\psi^+(x_j),\psi(x_k):\mathcal{H}^{\otimes n}\to\mathcal{H}^{\otimes n}, j,k=\overline{1,n}$, satisfying the anti-commutation relationships

$$\{\psi(x_j),\psi(x_k)\} = 0 = \{\psi^+(x_j),\psi^+(x_k)\}, \qquad (2.13)$$

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$\{\psi(x_j),\psi^+(x_k)\}=\delta_{jk},$

where I have redefined the anti-commutator notation $[\cdot,\cdot]_{+1}:=\{\cdot,\cdot\}$, and where $Cl_3(\sigma)$ is the Clifford algebra, generated by the basis generators of the $su(2)$ Lie algebra of the unitary symmetry group $SU(2)$, realized by the self-adjoint Pauli spin matrices $\sigma_j\in\mathrm{End}\ \mathbb{E}^2, j=\overline{1,3},$ where

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.14}$$

satisfying the following commutator relationships:

$$\sigma_1,\sigma_2]=2i\sigma_3,[\sigma_2,\sigma_3]=2i\sigma_1,[\sigma_3,\sigma_1]=2i\sigma_2, \tag{2.15}$$

or equivalently rewritten as

$$\sigma_j\sigma_k+\sigma_k\sigma_j=2\delta_{jk} \tag{2.16}$$

for all $j,k=\overline{1,3}.$ The Clifford algebra $Cl_{2n}(\psi,\psi^+),n\in\mathbb{N},$ is a well-known object in mathematical physics, which is constructed by means of the factorization procedure, the following way:

$$Cl_{2n}(\psi,\psi^+):=T_{2n}(\psi,\psi^+)/J_{2n}(\psi,\psi^+), \tag{2.17}$$

where $T_{2n}(\psi,\psi^+)-$ the tensor algebra over C spanned by the operator elements $\{\psi^+(x_j),\psi(x_k):\mathcal{H}^{\otimes n}\to\mathcal{H}^{\otimes n}:j,k=\overline{1,n}\},$ and $J_{2n}(\psi,\psi^+)-$ the two-sided ideal, generated by elements $\{\psi(x_k)\psi(x_j)+\psi(x_j)\psi(x_k),\psi(x_k)\psi^+(x_j)$ + $\psi^+(x_j)\psi(x_k)-\delta_{jk},\psi^+(x_k)\psi^+(x_j)+\psi^+(x_j)\psi^+(x_k):j,k=\overline{1,n}\},$ whose dimension $\dim Cl_{2n}(\psi,\psi^+)=2^{2n}.$ Concerning the Clifford algebra $Cl_3(\sigma)$ of the dimension $\dim Cl_3(\sigma)=2^3=8,$ it describes the internal degrees of freedom of electron, whose structure is based, to our regret, on a weakly known up-to-date physical nature of the corresponding internal quantum states, yet its Euclidean representation space $\mathbb{E}^2$ makes it possible to propose some useful inferences motivated by their interpretations. The following definition of the Clifford algebra representation.

Definition 2.1 *The left module of a Clifford algebra $Cl_m(\rho)$ is a linear finite-dimensional space L over C endowed with a linear representation mapping $\rho:Cl_m(\rho)\otimes L\to L,$ such that*

$$\rho(ab\otimes f)=\rho(a\otimes\rho(b\otimes f)) \tag{2.18}$$

for all $a,b\in Cl_m(\rho)$ and $f\in L$. In particular, a representation mapping $\rho:Cl_m(\rho)\otimes L\to L$ is a homomorphism of the corresponding algebras.

The algebraic relationships written above (2.15) should be naturally realized by means of basis operators of the canonical Clifford algebra $Cl_m(\beta,\beta^+)$, $m\in\mathrm{N}$, of the minimal dimension 4 = dimEnd E^2, generated by some virtual creation-annihilation operators $\beta,\beta^+:\Phi_{int}\to\Phi_{int},$ acting on the internal quantum states. That means that the following possible equivalence of representations of $Cl_3(\sigma)$ subject to the Clifford algebras $Cl_m(\beta,\beta^+),m\in\mathbb{N},$ looks as

$$\rho(Cl_3(\sigma))\simeq Cl_2(\beta,\beta^+)\oplus Cl_2(\beta,\beta^+), \tag{2.19}$$

That is m = 2 As a consequence, one derives the existence of some algebraic relationships $\sigma_j=\sigma_j(\beta,\beta^+),j=\overline{1,3},$ which can be found in the following way: first, one constructs the so-called "*chirality*" operator

$$\Gamma:=\exp(i\pi\beta^+\beta):\Phi_{int}\to\Phi_{int}, \tag{2.20}$$

which satisfies, since the image $\mathrm{Im}\left(\beta^+\beta\right)\simeq\mathbb{Z}_+,$ the evident condition $\tilde{A}^2=I$, and anti-commutes with operators $\beta,\beta^+:$

$$\{\Gamma,\beta\}=0=\{\Gamma,\beta^+\}. \tag{2.21}$$

The latter makes it possible to determine the following projection operators:

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$$P_{\pm} := (I \pm \Gamma)/2, \tag{2.22}$$

which splits the internal Fock space Φ_{int} into the direct sum: $\Phi_{int} = \Phi_{int}^{(+)} \oplus \Phi_{int}^{(-)}$, where $\Phi_{int}^{(\pm)} := P_{\pm}\Phi_{int}$. Moreover, from the condition $(\beta^+ + \beta)^2 = I$ the following transposition condition

$$(\beta^+ + \beta)\Phi_{int}^{(+)} = \Phi_{int}^{(-)} \tag{2.23}$$

holds. Having now observed that $\Gamma = I + i\pi\beta^+\beta - \pi^2\beta^+\beta\beta^+\beta + ... = I - 2\beta^+\beta \in Cl_2(\beta, \beta^+)$, one easily obtains that the operator representation expressions

$$\rho(\sigma_1) = (\beta^+ + \beta)/2, \rho(\sigma_2) = -i(\beta^+ - \beta)/2, \rho(\sigma_3) = -\Gamma \tag{2.24}$$

satisfy the determining commutator relationships (2.15). Now we remark that for the constructed above Clifford algebra $Cl_3(\sigma)$ representation, one has the natural equality $\dim Cl_3(\sigma) = 8 = \dim(\mathrm{End}\,\mathbb{E}^2 \otimes \mathbb{C}[\Gamma])$, giving rise to the next dimension equality dim $\Phi_{int} = \dim(\Phi_{int}^{(+)} \oplus \Phi_{int}^{(-)}) = 4$. The latter gives rise to the related representation of the $su(2)$ Lie algebra on the internal Fock space $\Phi_{int} = \Phi_{int}^{(+)} \oplus \Phi_{int}^{(-)}$ by means of the matrix vector-operator

$$\alpha := \{\alpha_j : \alpha_j = \sigma_1 \otimes \sigma_j \in \rho(Cl_3(\sigma)), j = \overline{1,3}\} \in \mathbb{E}^3 \otimes \mathrm{End}\mathbb{E}^4, \tag{2.25}$$

acting on the space $\mathbb{E}^4 \simeq \mathbb{E}^2 \times \mathbb{E}^2$ of 4 - spinors, where the symmetric matrices $\alpha_j \in$ End $\alpha_j \in \mathrm{End}\left(\mathbb{E}^2 \times \mathbb{E}^2\right), j = \overline{1,3}$, satisfy [3] the algebraic relationships similar to those of (2.16):

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} \tag{2.26}$$

for all $j,k = \overline{1,3}$. It is clear that the set of spin α - matrices above, generating the eight-dimensional Clifford algebra $Cl_3(\alpha)$, coincides with the corresponding represe tation of the naturally extended Clifford algebra $Cl_3(\sigma) \oplus \mathbb{C}[\epsilon]$ on the spinor space $\mathbb{E}^4 \simeq \Phi_{int} = \Phi_{int}^{(+)} \oplus \Phi_{int}^{(-)}$, where, by definition, $\epsilon = \sqrt{-i\sigma_1\sigma_2\sigma_3}$ is an operator, equivalent to the "*chirality*" operator $\Gamma \in Cl_2(\beta, \beta^+)$, considered above, pointing out that there exist exactly two non-equivalent representations of the Clifford algebra $Cl_3(\sigma)$ on the space $\Phi_{int} = \Phi_{int}^{(+)} \oplus \Phi_{int}^{(-)}$, namely the representation by the α- operators on $\mathbb{E}^4 \simeq \mathbb{E}^2 \times \mathbb{E}^2$.

Now, based on the reasonings above, we can state that constructed above the irreducible representation $\rho : Cl_3(\sigma) \to \mathrm{End}\left(\Phi_{int}^{(+)} \oplus \Phi_{int}^{(-)}\right)$ of the Clifford algebra $Cl_3(\sigma)$ is equivalent to that by means of the "*true*" symmetry spin vector-matrix $S \in \mathbb{E}^3 \otimes \mathrm{End}\,\mathbb{E}^4$, naturally reflecting the internal symmetry of electron quantum states, not depending on spatial [3,8] coordinates, but only on the related "*virtual*" degrees of freedom. As a result, one can see that the internal symmetry group $SU(2)$ generators have made it possible to realize the irreducible Clifford algebra $Cl_3(\sigma)$ representation $\rho : Cl_3(\sigma) \to \mathrm{End}\,\mathbb{E}^4$ on the spinor space $\mathbb{E}^4 \simeq \mathbb{E}^2 \times \mathbb{E}^2$ of the one-electron quantum states.

With now recall that about the constructed above representation $\rho : Cl_3(\sigma) \to \mathrm{End}\,\mathbb{E}^4$, the projections $S_p^{(\pm)} = \int_{\mathbb{R}^3} dx\psi^+\langle S \mid p\rangle\psi\mid_{\Phi_{int}^{(\pm)}}$ of the corresponding spin operator

$$S := \{\frac{\hbar}{2}S_j : S_j = I_2 \otimes \sigma_j \in \rho(Cl_3(\sigma)), j = \overline{1,3}\} \tag{2.27}$$

The electron momentum operator $p := \frac{\hbar}{i}\nabla_x, x \in \mathbb{R}^3$, when considered on the subspaces $\Phi_{int}^{(\pm)}$, are conserved in time, which was experimentally [3,8,11] confirmed. As we will demonstrate below, the latter can be effectively used for constructing the Hamiltonian operator for an electron within the picture devised above.

Let us take now into account that the *time translation symmetry generator*, represented by the corresponding Hamiltonian operator $H_f : \Phi^{el} \to \Phi^{el}$ on the corresponding Fock space Φ^{el}, which is generated by its finite-particle reductions $H_{f,n} : \Phi_n^{el} \to \Phi_n^{el}$, $n \in \mathrm{N}$, belonging to the representation of the Clifford algebra $Cl_{2n}(\psi, \psi^+) \otimes \rho(Cl_3(\sigma))$, $n \in \mathrm{N}$, as well as recall that the temporal evolution of any operator observable $A : \Phi^{el} \to \Phi^{el}$ with respect to the parameter $t \in \mathbb{R}$ is described [3,8,14] by means of the classical Heisenberg equation

$$\partial A / \partial t = 1/\hbar H_f, A]. \tag{2.28}$$

The operator commutation condition $[H_f, A]$ simply means that the observable operator quantity $A : \Phi^{el} \to \Phi^{el}$ on quantum

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states is conserved. Recalling now the projection operator relationships (2.23) and (2.24) jointly with the mentioned above experimentally confirmed physical property: the operator *spin projections* .

$$S_p^{(\pm)} = \int_{\mathbb{R}^3} dx\psi^+(x)\langle S \mid p\rangle\psi\,|_{\Phi\otimes\Phi_{int}^{(\pm)}} \quad = \frac{\hbar}{2}\int_{\mathbb{R}^3} dx\psi^+(x)\langle I_2 \otimes \sigma \mid p\rangle\psi\,|_{\Phi\otimes\Phi_{int}^{(\pm)}} =$$

$$= (\sigma_1 \otimes I_2)\int_{\mathbb{R}^3} dx\psi^+(x)\langle I_2 \otimes \sigma \mid p\rangle\psi\,|_{\Phi\otimes\Phi_{int}^{(\pm)}} = \frac{\hbar}{2}\int_{\mathbb{R}^3} dx\psi^+(x)\langle \alpha \mid p\rangle\psi\,|_{\Phi\otimes\Phi_{int}^{(\pm)}} \tag{2.29}$$

on the electron momentum $p := \frac{\hbar}{i}\nabla$ are conserved, that is

$$H_f, S_p^{(\pm)}]_{\Phi\otimes\Phi_{int}^{(\pm)}} = 0, \tag{2.30}$$

One can naturally derive the simplest quantum Hamiltonian operator expression as $H_f \simeq (\sigma_1 \otimes I_2) S_p \in \mathrm{End}(\Phi\otimes\Phi_{int})$ modulo a constant operator from the Clifford algebra $\rho(Cl_3(\sigma))$ representation of observable operators, that is

$$H_f = \int_{\mathbb{R}^3} dx\psi^+(x)\big(\hbar/(2i)\langle \alpha \mid \vec{\nabla}_x\rangle + m\beta\big)\psi \tag{2.31}$$

on the whole Fock space $\Phi^{el} = \Phi\otimes\Phi_{int}$, where $m \in \mathbb{R}_+$ denotes a so-called "*mass*" parameter, $\beta = \sigma_3 \otimes I, [\beta, S] = 0,$ *a priori* satisfying the conditions (2.30) and exactly coinciding [3] with the classical Dirac Hamiltonian. Similarly, the trivial condition $[H_f, N] = 0$ means that the number operator (2.12) is also conserved, meaning that the finite-particle Fock subspace $\Phi_n, n \in \mathbb{N},$ is invariant with respect to the time evolution.

Remark 2.2 *As was noted in* [43], *the appearance of the "mass" parameter in the Dirac Hamiltonian expression (2.31) in no way is physically motivated by the internal SU*(2) *symmetry of the electron discussed in detail above, but may rather be related to some virtual mechanism of the SU*(2)*- Higgs type symmetry breaking, as it was developed within the standard Weinberg-Salam hadron model. However, the electron as a lepton particle does not fit into this description scheme, so another approach to its resolution is required.*

Remark 2.3 *The electron spin S is, in some sense, modelled by means of a virtual space of internal parameters* $Z_{int}^3 \simeq \mathbb{R}^3$ *and its internal "momentum" operator* $p_{int} := \frac{\hbar}{i}\nabla_z, z \in Z_{int}^3,$ *in the internal microworld* Z_{int}^3 *as follows:* $S = z \times p_{int}$, *whose components satisfy the classical relationships*

$$S_1, S_2] = 2i\hbar S_3, [S_2, S_3] = 2i\hbar S_1, [S_3, S_1] = 2i\hbar S_2, \tag{2.32}$$

which imitate those for the classical angular momentum of an electron in the real space $\mathbb{R}^3$, yet nothing more.

As it was observed still by Dirac, the formal algebraic sum of the spin $I \otimes S : \Phi\otimes\Phi_{int} \to \Phi\otimes\Phi_{int}$ and the angular momentum $J \otimes I : \Phi\otimes\Phi_{int} \to \Phi\otimes\Phi_{int}$ operators also remains invariant in time, that is

$$H_f, J \otimes I + I \otimes S] = 0. \tag{2.33}$$

Turning back to the electron Hamiltonian operator (2.31), reduced on the finite-particle subspace $\Phi_n \otimes \Phi_{int}, n \in \mathbb{N},$ one obtains that it belongs to some representation of the Clifford algebra $Cl_{2n}(\psi,\psi^+) \otimes \rho(Cl_3(\sigma))$, for which the following invariance relationships

$$S_1 : \Phi_n \otimes \Phi_{int}^{(\pm)} \to \Phi_n \otimes \Phi_{int}^{(\mp)} \tag{2.34}$$

for all $n \in \mathrm{N}$ hold. The above means that, in general, the reduced electron Hamiltonian operators $H_{f,n} : \Phi_n \otimes \Phi_{int} \to \Phi_n \otimes \Phi_{int}$ for arbitrary number of particles $n \in \mathrm{N}$ act, respectively, as operators on the direct sums $\Phi_n \otimes \big(\Phi_{int}^{(+)} \oplus \Phi_{int}^{(-)}\big) \simeq \Phi_n \otimes \mathbb{E}^4,$ $n \in \mathrm{N}$ what is in complete agreement with the classical Dirac result [3] for the electron Hamiltonian operator (2.31). Taking into account analytical properties of discussed above quantum states, associated to electron and describing its evolution in the ambient Minkowski spacetime, one can draw an important conclusion that the electron spin S is a physical quantity, responsible for its *hidden internal structure*, mathematically related with the symmetry group $SU(2)$ of the electron quantum states and their representations, compatible with the related many–particle representations of the algebra of observable operators,

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generated by the basic Clifford algebra $Cl_{2n}(\psi,\psi^{+})\otimes\rho(Cl_3(\sigma))$ on the Fock subspaces $\Phi_n\otimes\Phi_{int}, n\in\mathbb{N}$, which is responsible for the description of quantum electron states by means of antisymmetric functions from the invariant Fock subspace Φ_n for all $n\in$ N, and which proves to be effective for studying the energy spectrum of electron, naturally interacting with the ambient quantum electromagnetic field. Conclusion

We reanalyzed the electron spin structure and its deep connection with the symmetry properties of the related representations of the basic Clifford algebra, generated by creation-annihilation operators on the Fock space. Making use of the special Clifford algebra representation corresponding to the Pauli $su(2)$– symmetry algebra generators, their chirality symmetry and the related temporal conservation of the spin projection on the electron momentum, there was derived a quantum Hamiltonian operator on the Fock space, whose finite-dimensional invariant projection coincides exactly with the classical Dirac operator, whose Lorentz invariance follows as a natural consequence. Its Lorentz invariance was stated as a natural consequence of the construction presented in the work [44-51].

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