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Review Article

Optical Solitons Solutions in Birefringent Fibers with Perturbed Generalized Gerdjikov–Ivanov Model using the Addendum to Sub-ODE Method

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Abstract

For the perturbed generalized Gerdjikov-Ivanov (GI) model in polarization-preserving fibers, we study the propagation properties of optical solitons in birefringent fibers. A complementary and effective integration method, namely an addendum to the Sub-ODE method, is used to analytically handle the coupled (GI) equations once they have been reduced to tractable forms via the use of an appropriate similarity transformation. By using this method, a wide range of accurate closed-form solutions may be created, such as bright, dark, kink-shaped, solitary, bell-shaped, straddled, Jacobi elliptic and Weierstrass elliptic doubly periodic waveforms. We carefully construct explicit parametric constraints regulating the existence of each class. The findings give important information for developing sophisticated nonlinear fibre-optic systems, engineering photonic crystal structures and enhancing the propagation control of ultrashort optical pulses in high-capacity communication networks, in addition to enhancing the analytical solution space of the (GI) model.

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1. Introduction

The nonlinear propagation of ultra-short optical pulses in single-mode and birefringent fibers is commonly described by higher-order generalizations of the nonlinear Schrödinger equation. Among these models, the Gerdjikov-Ivanov (GI) equation plays a central role because it incorporates quintic nonlinearity and nonlinear dispersion while still admitting exact analytical soliton solutions. As a result, the GI framework has been used to model a wide range of nonlinear wave phenomena, including optical pulses in fibers, rogue waves, plasma waves and matter-wave condensates (see, for example, Refs. [1-5] for foundational developments of the GI equation and related higher-order nonlinear Schr"odinger-type models, [6-10] for further analytical studies of soliton structures, modulation instability and higherorder nonlinear effects, and [11-15] for extensions, numerical investigations and applications to optical fiber systems, plasma waves and matter-wave condensates).

In realistic fiber-optic links, several additional physical effects must be taken into account, such as chromatic dispersion (CD), higher-order dispersion, self- and cross-phase modulation (SPM/XPM), nonlinear dispersion, intermodal dispersion (IMD), self-steepening (SS) and other perturbations. These higher-order contributions can significantly influence soliton stability, pulse shaping and interaction dynamics, especially in the presence of birefringence and polarization coupling, as reported in Refs [16-19].

In birefringent or polarization-preserving fibers, the optical field must be described by a coupled system for the two orthogonal polarization components. Generalized GI-type models for such media have been used to investigate coupled bright-bright and dark-dark solitons, polarization instabilities and energy exchange between modes. However, most existing works focus either on scalar GI equations or on coupled systems with a restricted set of higher-order effects, and only a limited number of explicit solution families are available for the perturbed generalized GI model that simultaneously incorporates CD, SPM/XPM, nonlinear dispersion, IMD and SS.

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To the best of our knowledge, there is still a lack of systematic analytical studies of optical solitons in birefringent fibers governed by the perturbed generalized GI model considered in this paper. In particular, explicit parametric conditions guaranteeing the existence of different soliton types (bright, dark, kink-shaped, singular, bell-shaped and doubly periodic waveforms) for the coupled system have not been fully characterized.

The main objective of this work is to construct and classify a broad family of exact optical soliton solutions for the perturbed generalized GI model in polarization-preserving birefringent fibers. After reducing the coupled partial differential equations to ordinary differential equations through an appropriate similarity transformation, we apply an addendum to the Sub-ODE method to obtain closed-form solutions [17-19]. The main contributions of this paper are as follows we formulate a perturbed generalized GI model for birefringent fibers and clarify the physical roles of all relevant coefficients; we derive a reduced ordinary differential equation describing the common amplitude profile of the two polarization components and express its coefficients in terms of the original physical parameters; we apply the addendum to the Sub-ODE method to obtain multiple classes of exact solutions, including bright, dark, kink-shaped, singular, bell-shaped, straddled, Jacobi elliptic, and Weierstrass elliptic doubly periodic solitons; and for each solution class we state the parametric constraints ensuring its existence and discuss their implications for controlling ultrashort pulse propagation in high-capacity birefringent fiber links. Taken together, these results broaden the analytical solution space of the GI model and provide practical guidance for the design of advanced nonlinear fiber optic and photonic crystal systems.

2. Governing model

For the generalized Gerdjikov-Ivanov (GI) model in polarization-preserving fibers [1,2], the dimensionless form is expressed as follows:

$$iq_t + aq_{xx} + b|q|^4 q + icq^2 q_x^* = i\left[\alpha q_x + \lambda\left(|q|^{2m} q\right)_x + \mu\left(|q|^{2m}\right)_x q + \theta |q|^{2m} q_x\right], \tag{1}$$

so that $q(x, t)$ is a complex-valued function representing the wave profile, and q^* is its complex conjugate, and $i=\sqrt{-1}$ the first term is the linear temporal evolution of solitons, and the coefficient of a gives the chromatic dispersion (CD), b is the quintic form of nonlinearity, c the nonlinear dispersion, α the inter-modal dispersion (IMD), λ the coefficient of self-steepening (SS) for short pulses, while and θ are the higher-order dispersion effects. Here m is the nonlinearity parameter.

For the first time in birefringent fibers, Eq. (1) splits into two halves as:

$$iU_t + a_1 U_{xx} + \left(b_1|U|^4 + c_1|U|^2|V|^2 + d_1|V|^4\right)U + i\left(e_1U^2 + f_1V^2\right)U_x^*$$
$$= i\ \left[\alpha_1 Ux + \lambda_1\left(|U|^{2m} U\right)_x + \mu_1\left(|U|^{2m}\right)xU + \theta_1|U|^{2m} U_x\right], \tag{2}$$

and

$$iV_t + a_2 V_{xx} + \left(b_2|V|^4 + c_2|V|^2|U|^2 + d_2|U|^4\right)V + i\left(e_2V^2 + f_2U^2\right)V_x^*$$
$$= i\ \left[\alpha_2 Vx + \lambda_2\left(|V|^{2m} V\right)x + \mu_2\left(|V|^{2m}\right)xV + \theta_2|V|^{2m} Vx\right], \tag{3}$$

where $U(x,t)$ and $V(x,t)$ are complex-valued functions that represents the wave profiles with $i=\sqrt{-1}$. while $a_j, b_j, c_j, d_j, e_j, f_j, \alpha_j, \lambda_j, \mu_j, \theta_j (j=1,2)$ are all real-valued constants. a_j is the coefficient of the chromatic dispersion (CD), b_j is the coefficient of the self-phase modulation (SPM), c_j, d_j are the coefficient of the crossphase modulation (XPM), e_j, f_j are the coefficient of the nonlinear dispersion terms, α_j is the coefficient of the inter-model dispersion (IMD), λ_j is the coefficient of the self-steepening (SS), μ_j and θ_j are the coefficient of the higher-order dispersion.

In order to find dark soliton solutions, singular soliton solutions, bell-shaped soliton solutions, kink-shaped soliton solutions, Jacobi elliptic doubly periodic type soliton solutions, Weierstrass elliptic doubly periodic type solutions, bright soliton solutions and straddled soliton solutions, this paper aims to solve Eqs. (2) and (3) using an addendum to the SubODE approach.

The following is how this article is structured: Section 2 provides the guiding model. A mathematical analysis is provided in section 3. We use an addendum to the Sub-ODE approach in part 4. Additional findings are presented in section 5 . Conclusions are finally shown in section 6.

3. Mathematical analysis

We assume the following forms for the wave profiles to solve Eqs.(2) and (3):

$$U\quad (x,t) = \phi_1(\xi)e^{i(-\kappa x+\omega t+\theta_0)},$$
$$V\quad (x,t) = \phi_2(\xi)e^{i(-\kappa x+\omega t+\theta_0)},$$
$$\xi\quad = x-\rho t, \tag{4}$$

where $\phi_j(\xi)(j=1,2)$ represent the amplitude components of wave transformation, ρ, θ_0, ω and k signify the velocity, the phase constant, the wave number and the frequency respectively. Inserting Eq.(4) into Eqs.(2) and (3), we get:

$$\phi_1\left[-\omega - a_1\kappa^2 - \kappa\alpha_1\right] + i\phi_1^{'}\left[-\rho - 2a_1\kappa - \alpha_1\right] + a_1\phi_1^{''} + b_1\phi_1^5 + c_1\phi_1^3\phi_2^2 + d_1\phi_2^4\phi_1$$
$$-\kappa e_1\phi_1^3 + ie_1\phi_1^2\phi_1^{'} - \kappa f_1\phi_1\phi_2^2 + if_1\phi_1^{'}\phi_2^2 - (2m+1)\lambda_1 i\phi_1^{2m}\phi_1^{'} - \lambda_1\kappa\phi_1^{2m+1}$$
$$-2mi\mu_1\phi_1^{2m}\phi_1^{'} - \theta_1\kappa\phi_1^{2m+1} - i\theta_1\phi_1^{'}\phi_1^{2m} = 0, \tag{5}$$

and

$$\phi_2\left[-\omega - a_2\kappa^2 - \kappa\alpha_2\right] + i\phi_2^{'}\left[-\rho - 2a_2\kappa - \alpha_2\right] + a_2\phi_2^{''} + b_2\phi_2^5 + c_2\phi_2^3\phi_1^2 + d_2\phi_1^4\phi_2$$
$$-\kappa e_2\phi_2^3 + ie_2\phi_2^2\phi_1^{'} - \kappa f_2\phi_2\phi_1^2 + if_2\phi_2^{'}\phi_1^2 - (2m+1)\lambda_2 i\phi_2^2 m\phi_2^{'} - \lambda_2\kappa\phi_2^{2m+1}$$
$$-2mi\mu_2\phi_2^{2m}\phi_2^{'} - \theta_2\kappa\phi_2^{2m+1} - i\ \theta_2\phi_2^{'}\phi_2^{2m} = 0, \tag{6}$$

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Where $' = \frac{d}{d\xi}$.

Now, for the sake of simplicity, we set

$$\phi_2(\xi) = \chi\phi_1(\xi), \tag{7}$$

so that $\chi \neq 0$ and x its nonzero constant, transforming Eqs. (5) and (6) into :

$$\left[-\omega - a_1\kappa^2 - \kappa\alpha_1\right]\phi_1 + i\left[-\rho - 2a_1\kappa - \alpha_1\right]\phi_1' + \left[b_1 + c_1\chi^2 + d_1\chi^4\right]\phi_1^5 + \kappa\left[-e_1 - f_1\chi^2\right]\phi_1^3$$

$$+a_1\phi_1'' + \kappa\left[-\lambda_1 - \theta_1\right]\phi_1^{2m+1} + i\left[e_1 + f_1\chi^2\right]\phi_1^2\phi_1' + i\left[-(2m+1)\lambda_1 - 2m\mu_1 - \theta_1\right]\phi_1^{2m}\phi_1' = 0\ , \tag{8}$$

and

$$\left[-\omega - a_2\kappa^2 - \kappa\alpha_2\right]\chi\phi_1 + i\chi\left[-\rho - 2a_2\kappa - \alpha_2\right]\phi_1' + a_2\chi\phi_1'' + \left[b_2\chi^4 + c_2\chi^2 + d_2\right]\chi\phi_1^5$$

$$+i\left[e_2\chi^3 + f_2\chi\right]\phi_1^2\phi_1' + \kappa\left[-e_2\chi^3 - f_2\chi\right]\phi_1^3 + \kappa\left[-\lambda_2\chi^{2m+1} - \theta_2\chi^{2m+1}\right]\phi_1^{2m+1}$$

$$+i\left[-(2m+1)\lambda_2\chi^{2m+1} - 2m\mu_2\chi^{2m+1} - \theta_2\chi^{2m+1}\right]\phi_1^{2m}\phi_1' = 0\ . \tag{9}$$

Then from Eqs. (8) and (9) yields the imaginary parts:

$$\left[-\rho - 2a_1\kappa - \alpha_1\right]\phi_1' + \left[e_1 + f_1\chi^2\right]\phi_1^2\phi_1' + \left[-(2m+1)\lambda_1 - 2m\mu_1 - \theta_1\right]\phi_1^{2m}\phi_1' = 0, \tag{10}$$

and

$$\left[-\rho - 2a_2\kappa - \alpha_2\right]\phi_1' + \left[e_2\chi^2 + f_2\right]\phi_1^2\phi_1' + \chi^{2m}\left[-(2m+1)\lambda_2 - 2m\mu_2 - \theta_2\right]\phi_1^{2m}\phi_1' = 0. \tag{11}$$

By integrating Eqs. (10) and (11) and using the principle of linear independent, we obtain the velocity and the frequency of the soliton as:

$$\rho = -2a_j\kappa - \alpha_j\ ,$$
$$\kappa = \frac{\alpha_2 - \alpha_1}{2(a_1 - a_2)}\ , \tag{12}$$

along with the constraint

$$e_1 + f_1\chi^2 = 0, e_2\chi^2 + f_2 = 0 \quad \Rightarrow e_1e_2 = f_1f_2\ ,$$
$$(2m+1)\lambda_j + 2m\mu_j + \theta_j = 0 \tag{13}$$

where $j = (1,2)$, provided $\alpha_2 \neq \alpha_1, a_1 \neq a_2$. Now, the real parts are

$$\left[-\omega - a_1\kappa^2 - \kappa\alpha_1\right]\phi_1 + a_1\phi_1'' + \left[b_1 + c_1\chi^2 + d_1\chi^4\right]\phi_1^5 + \kappa\left[-e_1 - f_1\chi^2\right]\phi_1^3$$

$$+\kappa\left[-\lambda_1 - \theta_1\right]\phi_1^{2m+1} = 0\ , \tag{14}$$

and

$$\left[-\omega - a_2\kappa^2 - \kappa\alpha_2\right]\phi_1 + a_2\phi_1'' + \left[b_2\chi^4 + c_2\chi^2 + d_2\right]\phi_1^5 + \kappa\left[-e_2\chi^2 - f_2\right]\phi_1^3$$

$$+\kappa\left[-\lambda_2 - \theta_2\right]\chi^{2m}\phi_1^{2m+1} = 0 \tag{15}$$

For simplicity, let us put $m = 1$, we get:

$$\left[-\omega - a_1\kappa^2 - \kappa\alpha_1\right]\phi_1 + a_1\phi_1'' + \left[b_1 + c_1\chi^2 + d_1\chi^4\right]\phi_1^5 + \kappa\left[-e_1 - f_1\chi^2 - \lambda_1 - \theta_1\right]\phi_1^3 = 0\ , \tag{16}$$

and

$$\left[-\omega - a_2\kappa^2 - \kappa\alpha_2\right]\phi_1 + a_2\phi_1'' + \left[b_2\chi^4 + c_2\chi^2 + d_2\right]\phi_1^5 + \kappa\left[-e_2\chi^2 - f_2 - \lambda_2\chi^2 - \theta_2\chi^2\right]\phi_1^3 = 0. \tag{17}$$

Equations (16) and (17) are equivalent under the following restrictions:

$$\frac{\omega + a_1\kappa^2 + \kappa\alpha_1}{\omega + a_2\kappa^2 + \kappa\alpha_2} = \frac{e_1 + f_1\chi^2 + \lambda_1 + \theta_1}{e_2\chi^2 + f_2 + \lambda_2\chi^2 + \chi^2\theta_2} = \frac{b_1 + c_1\chi^2 + d_1\chi^4}{b_2\chi^4 + c_2\chi^2 + d_2} = \frac{a_1}{a_2} \tag{18}$$

Balancing $\phi_1''(\xi)$ with $\phi_1^5(\xi)$ in Eq.(16) gives the balance number $N = \frac{1}{2}$, we use the transformation:

$$\phi_1(\xi) = V^{\frac{1}{2}}(\xi), \tag{19}$$

where $V(\xi)$ is a new function. By putting Eq. (19) into Eq.(16), we obtain the Equation:

$$V'^2(\xi) - 2V(\xi)V''(\xi) + \Delta_1V^2(\xi) + \Delta_2V^3(\xi) + \Delta_3V^4(\xi) = 0 \tag{20}$$

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where
$$\Delta_1 = \frac{-4}{a_1}\left[-\omega - a_1\kappa^2 - \kappa\alpha_1\right],$$
$$\Delta_2 = \frac{-4\kappa}{a_1}\left[-e_1 - f_1\chi^2 - \lambda_1 - \theta_1\right], \tag{21}$$
$$\Delta_3 = \frac{-4}{a_1}\left[b_1 + c_1\chi^2 + d_1\chi^4\right],$$

provided $a_1 \neq 0$.

Next, we will use one of integration methods to get the solitons of Eqs. (2) and (3).

For clarity, we summarize the notation used in the reduction procedure and in the subsequent solution families. The complex-valued functions $U(x,t)$ and $V(x,t)$ denote the slowly varying envelopes of the two orthogonal polarization components in the birefringent fiber, while $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are their real amplitude profiles in the co-moving coordinate $\xi = x - \rho t$. The parameters ρ, κ, ω and θ_0 represent, respectively, the soliton velocity, wave number, carrier frequency and constant phase shift, and the constant $x \neq 0$ in (7) measures the relative amplitude of the two polarizations.

In the reduced ordinary differential equation for $V(\xi)$, the quantities Δ1, Δ2 and Δ3 are effective real coefficients determined by the physical parameters of the original model through their definitions above; they control the balance between dispersion and nonlinearity.

4. An addendum to Sub-ODE approach

The Sub-ODE method will be used [17,18], as suggested by Zayed, et al. to solve Eq. (20). In order to accomplish this, we assume that Eq.(20) has the following formal solution:

$$V(\xi) = \sum_{s=0}^{N} A_s[H(\xi)]^s \,, \tag{22}$$

where $A_s(s = 0,1,2,..,N)$ are constants, provided $A_N \neq 0$, while $H(\xi)$ is the solution of the auxiliary equation:

$$H'^2(\xi) = A\,H^{2-2p}(\xi) + B\,H^{2-p}(\xi) + C\,H^2(\xi) + D\,H^{2+p}(\xi) + E\,H^{2+2p}(\xi)\,, \tag{23}$$

where *A, B, C, D* and *E* are constants, while *p* is a positive integer. It is well known [17,18] that Eq. (23) has many particular solutions, which will be used throughout this section to find the optical soliton solutions of Eq. (20). If $D[V(\xi)] = N$ then $D\left[V'(\xi)\right] = N + p$, $D\left[V''(\xi)\right] = N + 2p$, thus $D\left[V^{(r)}(\xi)\right] = N + rp$ and $D\left[V^{(r)}(\xi)V^s(\xi)\right] = (s+1)N + pr$.

By balancing the highest derivative $V''(\xi)V(\xi)$ and nonlinear term $V^4(\xi)$ in Eq. (20), we have:

$$2N + 2p = 4N \Rightarrow N = p\,. \tag{24}$$

Now, if we choose $p = 1$, then $N = 1$. Thus, we can place the solution of Eq. (20) as follows:

$$V(\xi) = A_o + A_1 H(\xi)\,, \tag{25}$$

where A_o, A_1 are arbitrary real constants, $A_1 \neq 0$, and $H(\xi)$ satisfies the auxiliary equation:

$$H'^2(\xi) = A + B\,H(\xi) + C\,H^2(\xi) + D\,H^3(\xi) + E\,H^4(\xi), E \neq 0. \tag{26}$$

Compensation Eqs. (25) and (26) into Eq. (20), then we are combining all the transactions of $[H(\xi)]^l\left[H'(\xi)\right]^f, (l = 0,1,2,\ldots 4, f = 0,1)$ and put these coefficients equal to zero, we have the following system of algebraic equations:

$$\begin{cases} H^4(\xi) : \Delta_3 A_1^4 - 3A_1^2 E = 0, \\ H^3(\xi) : -4A_0A_1E + 4\Delta_3 A_0A_1^3 + \Delta_2 A_1^3 - 2A_1^2 D = 0, \\ H^2(\xi) : -3A_1A_0D + 3\Delta_2 A_0A_1^2 + 6\Delta_3 A_0^2A_1^2 + \Delta_1 A_1^2 - A_1^2 C = 0, \\ H(\xi) : -2A_1A_0C + 2\Delta_1 A_1A_0 + 3\Delta_2 A_1A_0^2 + 4\Delta_3 A_1A_0^3 = 0, \\ H^\circ(\xi) : -A_1A_0B + \Delta_1 A_0^2 + \Delta_2 A_0^3 + \Delta_3 A_0^4 + A_1^2 A = 0. \end{cases} \tag{27}$$

According to the articles [17,18], we study the following sets:

The substitution of the ansatz (22) together with the auxiliary equation into the reduced equation and the collection of coefficients of the powers of $H(\xi)$ and $H'(\xi)$ lead to the algebraic system above. Since these algebraic manipulations follow the standard steps of the Sub-ODE method and are straightforward to verify symbolically (for example, with Maple), we do not reproduce all intermediate calculations for each solution set. In the following, we only report the resulting parameter constraints and the associated exact solution families in order to avoid unnecessary repetition.

Set-1. Inserting $A = B = D = 0$, in algebraic equations (27) and using Maple, we get:

$$A_0 = A_0, \quad A_1 = A_1, C = \frac{-\Delta_3 A_0^2}{3}, \quad E = \frac{\Delta_3 A_1^2}{3}, \tag{28}$$

with constaint conditions:

$$\Delta_1 = \frac{5\Delta_3 A_0^2}{3},$$
$$\Delta_2 = -\frac{8\Delta_3 A_0}{3}. \tag{29}$$

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When $C>0$ and $E<o$, we get the bright soliton solution:

$$U(x,t)=\varepsilon\left[A_0\left(1+\operatorname{sech} A_0\sqrt{\frac{-\Delta_3}{3}}\xi\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{30}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0\left(1+\operatorname{sech} A_0\sqrt{\frac{-\Delta_3}{3}}\xi\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{31}$$

where $\Delta_3<0,\varepsilon=\pm1,A_0>0$. The solutions Eqs. (30), (31) are obtained under the restrictions (29).

Set-2. Substituting $B=D=0,A=\frac{eC^2}{E}$, in the above algebraic equations (27) and using Maple, we get :

$$A_0=A_0,\quad A_1=A_1,\quad C=\frac{-2\Delta_3A_0^2}{3},\quad E=\frac{\Delta_3A_1^2}{3}, \tag{32}$$

with constaint conditions:

$$\Delta_1=\frac{4}{3}\Delta_3A_0^2$$
$$\Delta_2=-\frac{8}{3}\Delta_3A_0. \tag{33}$$

We get the dark soliton solutions:

$$U(x,t)=\varepsilon\left[A_0\left(1+\tanh A_0\sqrt{\frac{\Delta_3}{3}}\xi\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{34}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0\left(1+\tanh A_0\sqrt{\frac{\Delta_3}{3}}\xi\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{35}$$

where $C<0,E>0,\Delta_3>0,A_o>0,\varepsilon=\pm1$. The solutions Eqs. (34), (35) are obtained under the restrictions (33).

Set-3. Substituting $B=D=0,A=\frac{eC}{E^2}$, in the above algebraic equations (27) and using Maple, we get:

$$A_0=A_0,\quad A_1^2=\frac{-EA_0^4}{C\left(eCA_1^2+EA_0^2\right)}, \tag{36}$$

with constaint conditions:

$$\begin{aligned}\Delta_1&=\frac{CA_1^2+6EA_0^2}{A_1^2}\\ \Delta_2&=\frac{-8EA_0}{A_1^2}\\ \Delta_3&=\frac{3E}{A_1^2}\end{aligned} \tag{37}$$

where e is a constant. We get the following solutions:

(I) When $e=\frac{m_1^2\left(m_1^2-1\right)}{\left(2m_1^2-1\right)^2}$ then $A=\frac{C^2m_1^2\left(m_1^2-1\right)}{E\left(2m_1^2-1\right)^2},0<m_1<1$, we get the Jacobi elliptic solution:

$$U(x,t)=\varepsilon\left[A_0+A_1\left(-\frac{Cm_1^2}{E\left(2m_1^2-1\right)}\right)^{\frac{1}{2}}\operatorname{cn}\left(\sqrt{\frac{C}{2m_1^2-1}}\xi,m_1\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{38}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(-\frac{Cm_1^2}{E\left(2m_1^2-1\right)}\right)^{\frac{1}{2}}\operatorname{cn}\left(\sqrt{\frac{C}{2m_1^2-1}}\xi,m_1\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{39}$$

where $C>0$, $E<0$. The solutions of Eqs. (38), (39) are obtained under the restrictions (37).

(II) When $e=\frac{\left(1-m_1^2\right)}{\left(2-m_1^2\right)^2}$ then $A=\frac{C^2\left(1-m_1^2\right)}{E\left(2-m_1^2\right)^2},0<m_1<1$, we get the Jacobi elliptic solutions:

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$$U(x,t)=\varepsilon\left[A_0+A_1\left(-\frac{C}{E\left(2-m_1^2\right)}\right)^{\frac{1}{2}}\text{dn}\left(\sqrt{\frac{C}{2-m_1^2}}\xi,m_1\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)},\tag{40}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(-\frac{C}{E\left(2-m_1^2\right)}\right)^{\frac{1}{2}}\text{dn}\left(\sqrt{\frac{C}{2-m_1^2}}\xi,m_1\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)},\tag{41}$$

where $C>0$, $E<0$. The solutions of Eqs. (40), (41) are obtained under the restrictions (37).

(III) When $e=\frac{m_1^2}{\left(m_1^2+1\right)^2}$ then $A=\frac{C^2m_1^2}{E\left(m_1^2+1\right)^2}, 0<m_1<1$, we get the Jacobi elliptic solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1\left(-\frac{Cm_1^2}{E\left(m_1^2+1\right)}\right)\text{sn}\left(\sqrt{-\frac{C}{m_1^2+1}}\xi,m_1\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)},\tag{42}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(-\frac{Cm_1^2}{E\left(m_1^2+1\right)}\right)\text{sn}\left(\sqrt{-\frac{C}{m_1^2+1}}\xi,m_1\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)},\tag{43}$$

where $C<0$ and $E>0$. All the solutions of Eqs. (42), (43) exist under the restrictions (37).

Set-4. Inserting $A=B=0$, in the algebraic equations (27) and using Maple, we get the following cases:

(I) When $C>0, E=\frac{D^2}{4C}-C$, we get following results:

$$A_0=A_0,\quad A_1=A_1,\quad D=\frac{2C\left(A_0+A_1\right)}{A_0},\tag{44}$$

with constaint conditions:

$$\Delta_1=\frac{C\left(6A_0+A_1\right)}{A_1},$$

$$\Delta_2=-\frac{4C\left(3A_0+A_1\right)}{A_0A_1},$$

$$\Delta_3=\frac{3C\left(2A_0+A_1\right)}{A_0^2A_1},\tag{45}$$

Now, we get the bright soliton solutions:

$$U(x,t)=\varepsilon\left[A_0+\frac{A_1}{\cosh\sqrt{C}\xi-\frac{D}{2C}}\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)},\tag{46}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+\frac{A_1}{\cosh\sqrt{C}\xi-\frac{D}{2C}}\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)},\tag{47}$$

where $C>0$.The solutions of Eqs. (46), (47) are existed under the restrictions (45).

(II) When $C>0, E>0, D^2=4CE$, we get the results:

$$A_0=A_0,\quad A_1=A_1,\quad C=\frac{\Delta_3A_0^2}{3},\quad E=\frac{\Delta_3A_1^2}{3},\tag{48}$$

with constaint conditions

$$\Delta_1=\frac{\Delta_3A_0^2}{3},$$

$$\Delta_2=-\frac{4\Delta_3A_0}{3},\tag{49}$$

So, we get the dark soliton solutions:

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$$U(x,t)=\varepsilon\left[A_0+\frac{A_1}{2}\sqrt{\frac{C}{E}}\left(1+\frac{1}{2}\tanh\sqrt{C}\xi\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{50}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+\frac{A_1}{2}\sqrt{\frac{C}{E}}\left(1+\frac{1}{2}\tanh\sqrt{C}\xi\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{51}$$

provided $C>0,E>0,\Delta_3>0,\varepsilon=\pm 1$. The solutions of Eqs. (50), (51) are existed under the restrictions (49).

Set-5. Substituting $A=B=0,D=\sqrt{4CE}$, in the above algebraic equations (27) and using maple, we get the following cases:

$$A_0=0,\quad A_1=A_1,\quad C=\Delta_1,\quad E=\frac{\Delta_2^2A_1^2}{16\Delta_1} \tag{52}$$

with constaint conditions:

$$\Delta_3=\frac{3\Delta_2^2}{16\Delta_3}, \tag{53}$$

where $C>0,E>0,\Delta_1>0$. Then, we get alot of soliton solutions as following:

(I) the dark soliton solutions:

$$U(x,t)=\varepsilon\left[\frac{2\Delta_1}{\Delta_2}\left(1+\frac{1}{2}\tanh\sqrt{\Delta_1}\xi\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{54}$$

and

$$V(x,t)=\varepsilon\left[\frac{2\Delta_1}{\Delta_2}\left(1+\frac{1}{2}\tanh\sqrt{\Delta_1}\xi\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{55}$$

where $\Delta_1>0$.

(II) The singular soliton solutions:

$$U(x,t)=\varepsilon\left[-\frac{2\Delta_1}{\Delta_2}\left(1+\frac{1}{2}\coth\sqrt{\Delta_1}\xi\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{56}$$

and

$$V(x,t)=\chi\left[-\frac{2\Delta_1}{\Delta_2}\left(1+\frac{1}{2}coth\sqrt{\Delta_1}\xi\right)\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{57}$$

where $\Delta_1>0$.

(III) The combo-bright-dark soliton solutions:

$$U(x,t)=\left[\frac{\Delta_1DA_1sech^2\frac{\sqrt{\Delta_1}}{2}\xi}{D^2-\frac{\Delta_2^2A_1^2}{16}\left[1+\varepsilon\tanh\frac{\sqrt{\Delta_1}}{2}\xi\right]^2}\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{58}$$

and

$$V(x,t)=\chi\left[\frac{\Delta_1DA_1\operatorname{sech}^2\frac{\sqrt{\Delta_1}}{2}\xi}{D^2-\frac{\Delta_2^2A_1^2}{16}\left[1+\varepsilon\tanh\frac{\sqrt{\Delta_1}}{2}\xi\right]^2}\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{59}$$

where $\Delta_1>0,\ DA_1>0$.

Also,

$$U(x,t)=\left[-\frac{A_1\Delta_1\operatorname{sech}^2\frac{\sqrt{\Delta_1}}{2}\xi}{D+\frac{\varepsilon\Delta_2A_1}{2}\tanh\frac{\sqrt{\Delta_1}}{2}\xi}\right]^{\frac{1}{2}}e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{60}$$

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and

$$V(x,t)=\chi\left[-\frac{A_1\Delta_1\operatorname{sech}^2\frac{\sqrt{\Delta_1}}{2}\xi}{D+\frac{\varepsilon\Delta_2A_1}{2}\tanh\frac{\sqrt{\Delta_1}}{2}\xi}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{61}$$

where $\Delta_1>0, A_1<0$.

(IV) The combo-singular soliton solutions:

$$U(x,t)=\left[\frac{A_1\Delta_1D\operatorname{csch}^2\frac{\sqrt{\Delta_1}}{2}\xi}{D^2-\frac{\Delta_2^2A_1^2}{16}\left[1+\coth\frac{\sqrt{\Delta_1}}{2}\xi\right]^2}\right]^{\frac{1}{2}}e^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{62}$$

And

$$V(x,t)=\chi\left[\frac{A_1\Delta_1D\operatorname{csch}^2\frac{\sqrt{\Delta_1}}{2}\xi}{D^2-\frac{\Delta_2^2A_1^2}{16}\left[1+\coth\frac{\sqrt{\Delta_1}}{2}\xi\right]^2}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{63}$$

Also,

$$U(x,t)=\left[\frac{A_1\Delta_1\operatorname{csch}^2\frac{\sqrt{\Delta_1}}{2}\xi}{D+\frac{\varepsilon}{2}\Delta_2A_1\coth\frac{\sqrt{\Delta_1}}{2}\xi}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\vartheta_0)}. \tag{64}$$

and

$$V(x,t)=\chi\left[\frac{A_1\Delta_1\operatorname{csch}^2\frac{\sqrt{\Delta_1}}{2}\xi}{D+\frac{\varepsilon}{2}\Delta_2A_1\coth\frac{\sqrt{\Delta_1}}{2}\xi}\right]^{\frac{1}{2}}e^{i(-kx+\omega t+\theta_0)}, \tag{65}$$

where $\Delta_1>0, A_1D>0$. All the solutions of Eqs. (54)-(65) are existed under the restrictions (53).

Set-6. Substituting $A=0, B=\frac{8C^2}{27D}, E=\frac{D^2}{4C}, A_1>0$ and $C<0$ in the above algebraic equations (27) and using maple, we get the results:

$$A_o=o,\ A_1=A_1,\ C=\Delta_1, \tag{66}$$

with constaint conditions:

$$\begin{aligned}\Delta_2&=\frac{2D}{A_1}\\ \Delta_3&=\frac{3D^2}{4A_1^2\Delta_1}\end{aligned} \tag{67}$$

Now, we get the following hyperbolic functions solutions:

$$U(x,t)=\left[-\frac{4A_1\Delta_1\tanh^2\sqrt{-3\Delta_1}\xi}{9D\left(3+\frac{\tanh\sqrt{-3\Delta_1}\xi}{6}\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{68}$$

and

$$V(x,t)=\chi\left[-\frac{4A_1\Delta_1\tanh^2\sqrt{-3\Delta_1}\xi}{9D\left(3+\frac{\tanh\sqrt{-3\Delta_1}\xi}{6}\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{69}$$

Also,

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$$U(x,t)=\left[\frac{4A_1\Delta_1\coth^2\sqrt{-3\Delta_1}\xi}{9D\left(3+\frac{\coth\sqrt{-3\Delta_1}}{6}\xi\right)}\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta_0)}\ , \tag{70}$$

and

$$V(x,t)=\chi\left[\frac{4A_1\Delta_1\coth^2\sqrt{-3\Delta_1}\xi}{9D\left(3+\frac{\coth\sqrt{-3\Delta_1}}{6}\xi\right)}\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta_0)}. \tag{71}$$

where $\Delta_1<0,\ DA_1>0$. The solutions of Eqs. (68)-(71) are existed under the restrictions (67).

Set-7. Inserting $B=D=0$, in the algebraic equations (27) and using Maple, we get:

$$A_0=A_0,\quad A_1=A_1,\quad A=\frac{-A_0^2\left(\Delta_3A_0^2+3C\right)}{3A_1^2},\quad E=\frac{\Delta_3A_1^2}{3}, \tag{72}$$

with constaint conditions:

$$\Delta_1=2\Delta_3A_0^2+C,$$
$$\Delta_2=\frac{-8\Delta_3A_0}{3}, \tag{73}$$

where $E>0, C>0$ and $\Delta_3>0$. We get the following four Weierstrass Elliptic functions solutions:

(I)

$$U(x,t)=\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\wp[(\xi),g_2,g_3]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{74}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\wp[(\xi),g_2,g_3]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{75}$$

(II)

$$U(x,t)=\varepsilon\left[A_0+A_1\left(\frac{A}{\wp[(\xi),g_2,g_3]-\frac{C}{3}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{76}$$

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(\frac{A}{\wp[(\xi),g_2,g_3]-\frac{C}{3}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{77}$$

so that the invariants g_2, g_3 of the Weierstrass elliptic function solutions of Eqs. (74)-(77) are given by

$$g_2=\frac{4C^2-12AE}{3}\text{and } g_3=\frac{4C\left(-2C^2+9AE\right)}{27}. \tag{78}$$

(III)

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{A}\left(\frac{6\wp[(\xi),g_2,g_3]+C}{3\wp'[(\xi),g_2,g_3]}\right)\right]^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{79}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{A}\left(\frac{6\wp[(\xi),g_2,g_3]+C}{3\wp'[(\xi),g_2,g_3]}\right)\right]^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{80}$$

where $A>0$.

(IV)

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$$U(x,t)=\varepsilon\left[A_0+A_1\left(\frac{\wp'[(\xi),g_2,g_3]}{2\sqrt{E\wp[(\xi),g_2,g_3]+C}}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{81}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(\frac{\wp'[(\xi),g_2,g_3]}{2\sqrt{E}\wp[(\xi),g_2,g_3]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{82}$$

where $E>0, A_1>0$ and so that the invariants g_2, g_3 of the Weierstrass elliptic function solutions of Eqs.(79)(82) are given by

$$g_2=\frac{C^2}{12}+AE \text{ and } g_3=\frac{C\left(36AE-C^2\right)}{216}.\tag{83}$$

All the solutions of Eqs. (74)-(83) are obtained under the restrictions (73).

Set-8. Substituting $B=D=0, A=\dfrac{5C^2}{36E}$, in the above algebraic equations (27) and using the maple, we get:

$$A_0=A_0,\quad A_1=A_1, C=\frac{-2\Delta_3 A_0^2}{5},\quad E=\frac{\Delta_3 A_1^2}{3},\tag{84}$$

with constaint conditions:

$$\Delta_1=\frac{8\Delta_3 A_1^2}{5},$$

$$\Delta_2=-\frac{8\Delta_3 A_0}{3},\tag{85}$$

where $E>0, \Delta_3>0, \varepsilon=\pm1$, then we have the Weierstrass elliptic function solution:

$$U(x,t)=\varepsilon\left[A_0+\sqrt{5}A_1C\left(\frac{\wp[(\xi),g_2,g_3]+\frac{C}{3}}{3\sqrt{E_{g'}[(\xi),g_2,g_3]}}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{86}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+\sqrt{5}A_1C\left(\frac{\wp[(\xi),g_2,g_3]+\frac{C}{3}}{3\sqrt{E}\wp'[(\xi),g_2,g_3]}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{87}$$

where the invariants g_2, g_3 of the Weierstrass elliptic function solutions (86), (87) are given by

$$g_2=\frac{2C^2}{9} \text{ and } g_3=\frac{C^3}{54},\tag{88}$$

The solutions of Eqs. (86),(87) are existed under the restrictions (85).

The constants A_0, A_1, A, B, C, D and E appearing in the Sub-ODE ansatz and in the auxiliary equation are real parameters determined by the algebraic system and govern the amplitude, width and shape of the resulting soliton solutions.

The parameter $m_1\in(0,1)$ denotes the modulus of the Jacobi elliptic functions, $\varepsilon=\pm1$ is a sign parameter, and $l_j(j=1,2,3)$ are the three real roots of the cubic equation $4y^3-g_2y-g_3=0$ associated with the Weierstrass elliptic function $\delta(\xi;g_2,g_3)$. The quantities g_2 and g_3 are the corresponding Weierstrass invariants. Unless otherwise stated, all coefficients and parameters are taken to be real.

Compared with other direct integration schemes, such as the various Kudryashov-type and simplest-equation methods applied to nonlinear wave models [3-7], the present addendum to the Sub-ODE approach offers a compact and systematic framework for constructing wide classes of exact solutions of the perturbed GI model. By working with the auxiliary equation (23) and the polynomial ansatz (22), all higher-order effects (quintic nonlinearity, nonlinear dispersion, inter-modal dispersion, self-steepening and higher-order dispersion) enter only through the effective coefficients Δ_1, Δ_2 and Δ_3 in (21). Once these coefficients are fixed, the algebraic system (27) simultaneously generates bright, dark, kink-shaped, solitary, bell-shaped, straddled, Jacobi elliptic and Weierstrass elliptic doubly periodic solutions from a single computational procedure, rather than requiring separate ad hoc ans"atze for each waveform family.

From this point of view, the addendum to the Sub-ODE method complements perturbative treatments of optical solitons, where higher-order terms are typically regarded as small corrections to an underlying NLSE or unperturbed GI soliton [2,14,19]. In the present formulation, the perturbation parameters are kept arbitrary and are absorbed into Δ_1, Δ_2 and Δ_3, so that the resulting soliton families remain valid even when the perturbations are not infinitesimally small. This non-perturbative feature is consistent with recent applications of the (modified) Sub-ODE framework to chirped and cubic-quartic solitons in related models [17,18], and it is particularly useful in regimes where stochastic effects or Kudryashov-type nonlinearities play a significant role.

From an implementation perspective, the addendum to the Sub-ODE approach is straightforward to use in practice. For any prescribed set of physical parameters in the birefringent fiber, one first computes the effective coefficients Δ_1, Δ_2 and Δ_3 via (21), substitutes them into the algebraic system (27), and then solves this system for the constants A_0, A_1, A, B, C, D and E with the aid of a symbolic package such as Maple or Mathematica. The corresponding parametric constraints single out the regions in the coefficient space where robust soliton profiles exist. This makes it possible to perform systematic parameter scans to design birefringent fiber configurations that support a desired soliton type (bright, dark, kink, Jacobi elliptic or Weierstrass elliptic) and to quantify how Kudryashov's nonlinearity and multiplicative noise modify the amplitude, width and phase of the optical pulses in realistic fiber-optic communication settings.

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5. Further results

It is well known [20,21] that, we can write the Weierstrass elliptic function $\wp(\xi;g_2,g_3)$ as follows:

$$\left.\begin{aligned}\wp(\xi;g_2,g_3)&=l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\\ \wp(\xi;g_2,g_3)&=l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\end{aligned}\right\},\tag{89}$$

in terms of the Jacobian elliptic functions where $m_1=\sqrt{\frac{l_2-l_3}{l_1-l_3}}$ is the modulus of the Jacobian elliptic function; $l_j(j=1,2,3),\ l_1\ge l_2\ge l_3$ are the three roots of the cubic equation $4y^3-g_2y-g_3=0$. Substituting Eq. (89) into Eqs. (74) and (75) we have Jacobi elliptic solutions:

$$U(x,t)=\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\right]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{90}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\right]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{91}$$

also,

$$U(x,t)=\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\right]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{92}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\right]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{93}$$

In particular, if $m_1\to 1$, then $l_1\to l_2$ and we get $\mathrm{cn}(\xi,1)\to\mathrm{sech}(\xi)$ and $\mathrm{ns}(\xi,1)\to\coth(\xi)$. Now, we get the bright soliton solutions:

$$U(x,t)=\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_1-l_3}\,\xi\right)\right]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{94}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_1-l_3}\,\xi\right)\right]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{95}$$

and the singular soliton solutions:

$$U(x,t)=\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_1-l_3}\,\xi\right)\right]-\frac{C}{3}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{96}$$

$$V(x,t)=\chi\varepsilon\left[A_0+\frac{A_1}{\sqrt{E}}\left(\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_1-l_3}\,\xi-\frac{C}{3}\right)\right]\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{97}$$

provided $E>0$.

Substituting Eq. (89) into Eqs. (76) and (77) we have Jacobi elliptic solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\right]-C}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{98}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\right]-C}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{99}$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\,\xi;m_1\right)\right]-C}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{100}$$

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and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]-C}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\qquad(101)$$

In particular, if $m_1\to1$, then $l_1\to l_2$ and we get $\mathrm{cn}(\xi,1)\to\mathrm{sech}(\xi)$ and $\mathrm{ns}(\xi,1)\to\coth(\xi)$. Now, we get the soliton solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_1-l_3}\xi\right)\right]-C}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\qquad(102)$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_1-l_3}\xi\right)\right]-C}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\qquad(103)$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_3+(l_2-l_3)coth^2\sqrt{l_1-l_3}\xi-C\right]}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-kx+\omega t+\theta_0)},\qquad(104)$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{3}\left(\frac{A}{3\left[l_3+(l_2-l_3)coth^2\sqrt{l_1-l_3}\xi\right]-C}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}.\qquad(105)$$

Substituting Eq. (89) into Eqs. (78) and (79), we get Jacobi elliptic solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{A}\left(\frac{6\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+C}{3\sqrt{l_1-l_3}\,(l_2-l_3)\,\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}\qquad(106)$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{A}\left(\frac{6\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\xi;m_{11}\right)\right]+C}{3\sqrt{l_1-l_3}\,(l_2-l_3)\,\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\qquad(107)$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{A}\left(-\frac{6\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+C}{3\sqrt{l_1-l_3}\,(l_1-l_3)\,\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\qquad(108)$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{A}\left(-\frac{6\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+C}{3\sqrt{l_1-l_3}\,(l_1-l_3)\,\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\qquad(109)$$

Provided $A>0$. In particular, if $m_1\to1$, then $l_1\to l_2$ and we get $\mathrm{cn}(\xi,1)\to\mathrm{sech}(\xi)$ and $\mathrm{ns}(\xi,1)\to\coth(\xi)$. Now, we get the combo bright-dark soliton solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{A}\left(\frac{6\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\right]+C}{3\sqrt{l_2-l_3}\,(l_2-l_3)\,\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\tanh\left(\sqrt{l_2-l_3}\xi\right)}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\qquad(110)$$

and

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$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{A}\left(\frac{6\left[l_2-(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\right]+C}{3\sqrt{l_2-l_3}\,(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\tanh\left(\sqrt{l_2-l_3}\xi\right)}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{111}$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1\sqrt{A}\left(-\frac{6\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\,\xi\right)\right]+c}{3\sqrt{l_2-l_3}\,(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\,\xi\right)\coth^3\left(\sqrt{l_2-l_3}\,\xi\right)}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{112}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\sqrt{A}\left(-\frac{6\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\,\xi\right)\right]+c}{3\sqrt{l_2-l_3}\,(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\,\xi\right)\coth^3\left(\sqrt{l_2-l_3}\,\xi\right)}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{113}$$

provided $A>0$.

Substituting Eq. (89) into Eqs. (80) and (81), we get Jacobi elliptic solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_1-l_3}\,(l_2-l_3)\operatorname{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{sn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)}{2\sqrt{E}\left[l_2-(l_2-l_3)\operatorname{cn}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{114}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_1-l_3}\,(l_2-l_3)\operatorname{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{sn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)}{2\sqrt{E}\left[l_2-(l_2-l_3)\operatorname{cn}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{115}$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_1-l_3}\,(l_1-l_3)\operatorname{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{ns}^3\left(\sqrt{l_1-l_3}\xi;m_1\right)}{2\sqrt{E}\left[l_3+(l_1-l_3)\operatorname{ns}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{116}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_1-l_3}\,(l_1-l_3)\operatorname{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\operatorname{ns}^3\left(\sqrt{l_1-l_3}\xi;m_1\right)}{2\sqrt{E}\left[l_3+(l_1-l_3)\operatorname{ns}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{117}$$

Provided $E>0$. In particular, if $m_1\to 1$, then $l_1\to l_2$ and we have $\operatorname{cn}(\xi,1)\to\operatorname{sech}(\xi)$ and $\operatorname{ns}(\xi,1)\to\coth(\xi)$. Now, we get the combo bright-dark soliton solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_2-l_3}\,(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\tanh\left(\sqrt{l_2-l_3}\xi\right)}{2\sqrt{E}\left[l_2-(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{118}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_2-l_3}\,(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\tanh\left(\sqrt{l_2-l_3}\xi\right)}{2\sqrt{E}\left[l_2-(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{119}$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_2-l_3}\,(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\coth^3\left(\sqrt{l_2-l_3}\xi\right)}{2\sqrt{E}\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\xi\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{120}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1\left(\frac{\sqrt{l_2-l_3}\,(l_2-l_3)\operatorname{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\coth^3\left(\sqrt{l_2-l_3}\xi\right)}{2\sqrt{E}\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\xi\right)\right]+C}\right)\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{121}$$

Substituting Eq. (89) into Eqs. (82) and (83), we get Jacobi elliptic solutions:

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$$U(x,t)=\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+\frac{C}{3}}{3\sqrt{l_1-l_3}(l_2-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{122}$$

$$V(x,t)=\chi\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+\frac{C}{3}}{3\sqrt{l_1-l_3}(l_2-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{123}$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+\frac{C}{3}}{3\sqrt{l_1-l_3}(l_1-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{124}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\xi;m_1\right)\right]+\frac{C}{3}}{3\sqrt{l_1-l_3}(l_1-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\xi;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\xi;m_1\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{125}$$

In particular, if $m_1\to 1$, then $l_1\to l_2$ and we have $\mathrm{cn}(\xi,1)\to\mathrm{sech}(\xi)$ and $\mathrm{ns}(\xi,1)\to\coth(\xi)$. Now, we get the combo bright-dark soliton solutions:

$$U(x,t)=\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\right]+\frac{C}{3}}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\tanh\left(\sqrt{l_2-l_3}\xi\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{126}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\right]+\frac{C}{3}}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\tanh\left(\sqrt{l_2-l_3}\xi\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{127}$$

also,

$$U(x,t)=\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_3+(l_2-l_3)\coth^2\left(\frac{V}{l_2-l_3}\xi\right)\right]+\frac{c}{3}}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\coth^3\left(\sqrt{l_2-l_3}\xi\right)}\right]^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{128}$$

and

$$V(x,t)=\chi\varepsilon\left[A_0+A_1C\sqrt{\frac{5}{E}}\frac{\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\xi\right)\right]+\frac{c}{3}}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\xi\right)\coth^3\left(\sqrt{l_2-l_3}\xi\right)}\right]^{\frac{1}{2}}e^{i(-kx+\omega t+\theta_0)}. \tag{129}$$

By adopting diverse parameter combinations for p and N, one can also obtain various soliton solutions of Eq. (16).

6. Modulation instability and soliton relevance

To gain insight into the robustness of the constructed solutions, we perform a modulation instability (MI) analysis of a constant-amplitude background associated with the coupled perturbed generalized GI model. For simplicity, we consider the conservative limit of Eqs. (2)-(3) in which the higher-order dissipative and driving terms on the right-hand side of Eq. (1) are neglected, and we focus on polarization-locked states consistent with the reduction $\varphi_2(\xi)=\chi\varphi_1(\xi)$ used in Section 3. In this case, the nonlinear coupling terms can be written in the form

$$\left(b_1|U|^4+c_1|U|^2|V|^2+d_1|V|^4\right)U=g_{\mathrm{eff}}|U|^4U,$$

where the effective nonlinear coefficient

$$g_{\mathrm{eff}}=b_1+c_1\chi^2+d_1\chi^4$$

collects the contributions of self- and cross-phase modulation between the two polarization components.

We first consider a continuous-wave (CW) background of the form

$$U(x,t)=U_oe^{i\beta t},\quad V(x,t)=\chi U_oe^{i\beta t}, \tag{1}$$

where U_0 is a constant complex amplitude. Substituting (1) into the conservative part of Eqs. (2)-(3) and using the fact that all x-derivatives vanish, we obtain the nonlinear frequency shift

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$\beta = -g_{\text{eff}} |U_o|^4$

To study the MI of this CW background, we add small perturbations with wavenumber K and frequency Ω according to

$$U(x,t) = [U_0 + u_1(x,t)]e^{i\beta t}, \quad V(x,t) = [\chi U_0 + v_1(x,t)]e^{i\beta t},$$

and take (u_1, v_1) to be superpositions of normal modes proportional to $e^{i(Kx-\Omega t)}$ and $e^{-i(Kx-\Omega t)}$. Restricting attention to perturbations that preserve the polarization ratio (i.e. $v_1 \approx \chi u_1$), which is natural in the present birefringent setting, the linearized system reduces to the well-known MI problem for a scalar higher-order NLSE-type model with nonlinearity $g_{\text{eff}} |U|^4 U$. In this case, the dispersion relation for the perturbations takes the standard form

$$\Omega^2(K) = a_1^2 K^4 - 4a_1 g_{\text{eff}} |U_0|^4 K^2 . \tag{2}$$

Using the definition of Δ_3 in (21), namely

$$\Delta_3 = -\frac{4}{a_1}\left(b_1 + c_1\chi^2 + d_1\chi^4\right) = -\frac{4}{a_1} g_{\text{eff}},$$

we can rewrite (2) as

$$\Omega^2(K) = a_1^2 K^2 \left(K^2 + \Delta_3 |U_0|^4\right). \tag{3}$$

Modulation instability occurs when there exist real values of K such that $\Omega^2(K) < 0$. From (3), this is possible if and only if $\Delta_3 < 0$, in which case the instability band is

$$o < K^2 < -\Delta_3 |U_o|^4,$$

and the corresponding MI growth rate is

$$\Gamma(K) = | Im\Omega(K) | = |a_1| K^p - K^2 - \Delta_3 |U_o|^4, \quad o < K^2 < -\Delta_3 |U_o|^4 .$$

If $\Delta_3 > 0$, then $\Omega^2(K) > 0$ for all K and the CW background is modulationally stable.

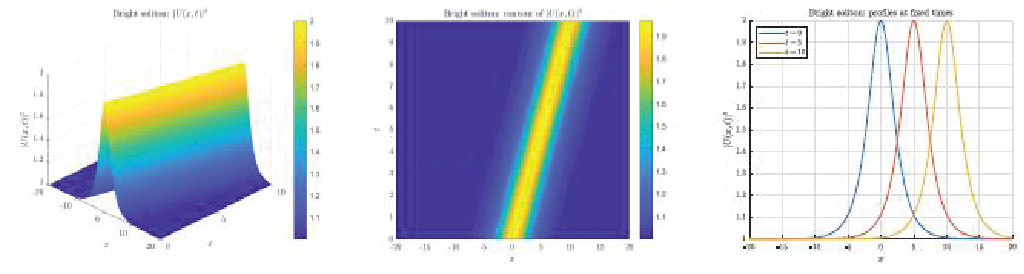
This MI criterion can be directly related to the parameter restrictions obtained in Section 5 for the different solution families. In particular, the bright and bell-shaped solitons constructed from the Sub-ODE method exist under conditions where $\Delta_3 < 0$, i.e. precisely in the modulationally unstable regime. In this case, the bright solitons may be interpreted as robust localized structures that saturate the MI of a CW or broad pulse background and correspond to localized spikes in the intensity of the two polarization components of the birefringent fiber. By contrast, the dark and kink-shaped soliton solutions are supported when $\Delta_3 > 0$ and the CW background is modulationally stable. These profiles describe persistent intensity dips or transition fronts on a stable background and are therefore relevant for normal-dispersion regimes and intensity-encoded channels in fiber optics.

The Jacobi and Weierstrass elliptic function solutions obtained in this work can be viewed as nonlinear wavetrains that interpolate between pure MI patterns and isolated soliton states. For parameter choices close to the boundary of the MI band, they describe almost-harmonic modulations of the background, whereas in the strongly nonlinear regime they represent trains of bright or dark pulses that can model pulse patterns in modelocked fiber lasers, dispersion-managed transmission lines and birefringent couplers. The fixed polarization ratio imposed by $\varphi_2(\xi) = \chi\varphi_1(\xi)$ implies that all these structures are polarization-locked, which is of practical interest for polarization-preserving links and all-optical switching devices in high-capacity fiber-optic systems.

We emphasize that including the full higher-order perturbation terms of Eq. (1) would quantitatively modify the MI gain spectrum (for example, by shifting the instability band or skewing it in wavenumber space), but the qualitative conclusion that the sign of Δ_3 controls the presence or absence of MI remains valid. A more detailed stability analysis of individual soliton profiles, based on spectral perturbation theory of the linearized operator, is an interesting problem but lies beyond the scope of the present work.

7. Results and discussion

In the revised manuscript we have supplemented the analytical results with three representative examples of the intensity profile $|U(x,t)|^2$ corresponding to a bright soliton, a dark (kink-type) soliton, and a Jacobi-elliptic periodic wave. These solutions are given by Eqs. (30), (34), and (38), respectively, and the associated profiles are displayed in Figures 1-3. For each case we show a three-dimensional surface plot in the (x, t) - plane, a contour map of the intensity, and one-dimensional cuts at selected propagation distances to illustrate the evolution of the pulse shape.



(a) Intensity $|U(x,t)|^2$ – 3D surface of the bright-soliton profile from Eq. (30). (b) Intensity – contour map corresponding to the bright-soliton solution in Eq. (30). (c) Profiles at selected times illustrating the temporal evolution of the bright-soliton intensity.

Figure 1: Profiles of the solution $U(x,t)$ from Eq. (30) demonstrating the spatial–temporal dynamics of the bright-soliton intensity, as depicted by surface, contour, and cross-sectional views.

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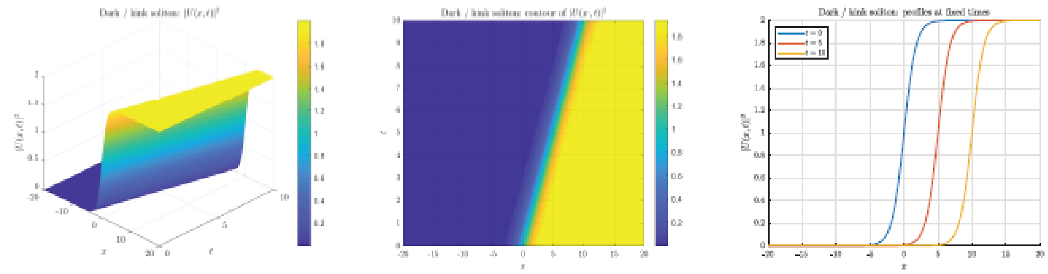
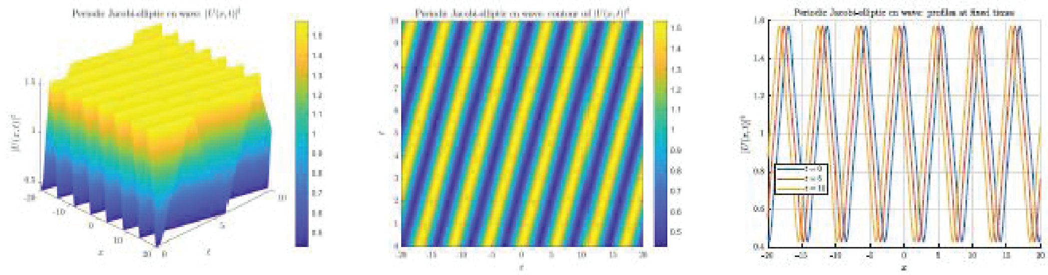


(a) Intensity $|U(x,t)|^2$ – 3D surface of the kink-type (dark) soliton from Eq. (34). (b) Intensity – contour map corresponding to the kink-type (dark) soliton in Eq. (34). (c) Profiles at selected times illustrating the propagation of the kink-type (dark) intensity front.

Figure 2: Profiles of the solution *U(x,t)* from Eq. (34) demonstrating the spatial-temporal dynamics of the kink-type (dark) soliton intensity, as depicted by surface, contour, and cross-sectional views.



(a) Intensity $|U(x,t)|^2$ – 3D surface of the Jacobi–elliptic cn–type periodic wave from Eq. (38). (b) Intensity – contour map corresponding to the Jacobi–elliptic cn–type periodic solution in Eq. (38). (c) Profiles at selected times showing successive peaks and troughs of the cn–type periodic intensity pattern.

Figure 3: Profiles of the solution *U(x,t)* from Eq. (38) demonstrating the spatial-temporal dynamics of the Jacobi elliptic cn-type periodic-wave intensity, as depicted by surface, contour, and cross-sectional views.

The parametric constraints guaranteeing the existence of these solutions follow directly from the addendum to the Sub-ODE method. After performing the similarity reduction (4) together with the polarization locking $\phi_2(\xi) = \chi\phi_1(\xi)$ in (7), the coupled system is reduced to the single real ODE (20) for $V(\xi)$, whose coefficients Δ_1, Δ_2 and Δ_3 are expressed in terms of the physical fiber parameters through Eq. (21). Substituting the polynomial ansatz (22)-(25) and the auxiliary equation (23) into (20), and requiring that all coefficients of the independent monomials in $H(\xi)$ and $H'(\xi)$ vanish, leads to the algebraic system (27). Solving this system for each solution set yields explicit relations between the auxiliary parameters A_0, A_1, A, B, C, D, E and the effective coefficients $\Delta_1, \Delta_2, \Delta_3$. In particular, the bright soliton (30) is obtained under the constraints (29), the dark (kink-type) soliton (34) under (33), and the Jacobi-elliptic periodic solution (38) under (37). Via Eq. (21), these relations translate into explicit inequalities for the chromatic-dispersion coefficient a_1, the self- and cross-phase modulation coefficients b_1, c_1, d_1, the higher-order coefficients $e_1, f_1, \lambda_1, \mu_1, \theta_1$, and the polarization amplitude ratio *X*. Hence, the existence domains of the bright, dark and periodic waves can be viewed as regions in the $(\Delta_1, \Delta_2, \Delta_3)$-space and, equivalently, in the space of physical fiber parameters.

Because Δ_1, Δ_2 and Δ_3 depend linearly on the underlying physical coefficients, the soliton characteristics (amplitude, background level, and width) respond smoothly to small parameter variations that keep the sign pattern and magnitude relations of Eqs. (29), (33) and (37) unchanged. In contrast, approaching the boundary of these constraint sets, and in particular the hypersurface $\Delta_3 = 0$, induces qualitative changes in the solution behavior. As discussed in the modulation-instability analysis preceding this section, the sign of Δ_3 controls whether a continuous-wave background is modulationally unstable or stable: $_3 < 0$ corresponds to an MIunstable regime supporting bright localized structures such as the solution in Eq. (30), whereas $\Delta_3 > 0$ yields a stable background on which dark solitons and periodic wavetrains, exemplified by Eqs. (34) and (38), can propagate robustly. The explicit constraints obtained from the Sub-ODE framework therefore provide a clear and quantitative link between the physical fiber parameters and the observed bright, dark and periodic dynamics illustrated in Figures 1-3.

In order to visualize the analytical solutions and to highlight the impact of the parametric constraints derived above, we now plot the intensity profiles associated with Eqs. (30), (34) and (38). Each figure consists of a threedimensional surface representation of $|(x, t)|^2$, a contour map in the (x, t)-plane, and several one- dimensional profiles at selected propagation distances. These complementary views make it possible to inspect the localization properties, background level and periodicity of the bright, dark and Jacobi-elliptic solutions as the field propagates along the birefringent fiber.

Figures 1-3 summarize the distinct propagation scenarios supported by the model. Figure 1 shows a localized bright pulse evolving on a vanishing background, with the surface and contour plots emphasizing the strong spatial-temporal localization of the intensity and the cross-sections revealing the gradual broadening of the pulse. In contrast, Figure 2 depicts a kink-type (dark) structure that connects two asymptotic intensity levels, illustrating how the phase and amplitude rearrange across a sharp transition layer while preserving the background. Finally, Figure 3 displays a Jacobi-elliptic cn-type periodic wavetrain, where the surface, contour and cross-sectional views highlight the regular sequence of intensity peaks and troughs along the propagation coordinate, providing a clear visualization of the underlying periodic dynamics.

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8. Conclusion

This paper discovered The Gerdjikov-Ivanov (GI) equation provides a powerful way to study solitons in realworld. This work shows how (GI) solitons behave under noise and strong nonlinearities, revealing their surprising stability. The results could improve optical communications and help control extreme waves by The addendum to Sub-ODE approach method were also used to discover bell-shaped soliton solutions, Jacobi elliptic doubly periodic type soliton solutions, kink-shaped soliton solutions, Weierstrass elliptic doubly periodic type solutions, singular soliton solutions, bright soliton solutions and straddled soliton solutions. Soliton solutions are produced by these techniques, and they are then given together with the pertinent existence conditions that are established by the parameter constraints. The paper's conclusions are incredibly inspiring and hopeful. The future looks very bright in light of these findings.

In the present work we have focused on the analytical construction and classification of exact soliton and periodic-wave solutions of the perturbed generalized GI model in birefringent fibers. A natural continuation of this study is to complement the analytical results with direct numerical simulations of pulse propagation. In particular, as future work we plan to implement split-step Fourier and finite-difference schemes for Eqs. (2) and (3), using realistic sets of optical fiber parameters (dispersion, nonlinearity and birefringence) in order to visualize the evolution of the obtained bright, dark, kink-shaped and elliptic soliton profiles, to assess their robustness in the presence of perturbations, and to quantify the impact of higher-order effects and noise. Such simulations will provide a detailed numerical confirmation of the analytical solutions reported here and will further clarify their practical applicability in fiber-optic communication and photonic devices.

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