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Mini Review

Uniqueness Analysis of Boundary Value Problems with Variable-order Caputo Fractional Derivatives

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Abstract

This work is devoted to a boundary value problem for variable-order Caputo fractional derivatives. The obtained conditions guarantee the uniqueness of solutions under given boundary conditions. The main tool in the analysis is the Banach contraction principle, with which one can assert that the problem admits exactly one solution. A numerical example is also presented for illustration and confirmation of the theoretical results.

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1. Introduction

Fractional calculus has developed as a helpful method for modeling complex dynamical systems where memory and hereditary properties are present [1,2]. In contrast to classical integer-order derivatives, the fractional-order derivatives involve nonlocal effects innately, which become so important in describing dynamics in materials, diffusion processes, and control systems [3,4]. The Caputo fractional-order derivative can be applied in practice since it allows one to include common initial and boundary conditions naturally [5]. More recently, VariableOrder Fractional Differential Equations(VO-FDE) have also been developed, for which the order of differentiation depends on time, space, or state, offering thus more flexibility when modeling processes whose dynamics evolve with time [6,7].

BVPs involving VO-FDE appear commonly in engineering and physical applications, such as anomalous diffusion, viscoelastic materials, and biological models [8,9]. These problems include non-linearities in the source term or boundary conditions that provide a nontrivial framework for proving the occurrence and singularity of solutions. Applications of fixed point methods, including the Banach contraction and Krasnoselskii theorems, present the basic instruments to establish rigorous results for such BVPs and provide explicit conditions under which solutions are guaranteed to exist and be unique [10,11].

Despite the substantial theoretical developments, numerical examples and practical validations remain essential to demonstrate the applicability of these results [12,13]. Studies on variable-order fractional BVPs must consider singularity and stability of solutions under perturbations to ensure reliability in real-world modeling [14,15]. Motivated by these considerations, the present work focuses on a class of Caputo variable-order fractional BVPs, establishing sufficient conditions for uniqueness and illustrating the theoretical findings through numerical examples.

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Motivated by prior studies on VO-FDEs [7,8,10]. Specifically, we consider the BVP

$$\begin{cases} {}^{C}D^{\mu(t)}u(t)+\alpha(t)u(t)=f(t,u(t)), for\, t\in[\rho_1,\rho_2] \\ u(\rho_1)+\left({}_{\rho_1}I^{\zeta}u\right)(\xi)=\beta \end{cases} \quad (1)$$

where ${}^{C}D^{\mu(t)}$ denotes the Caputo fractional derivative of variable order $\mu(t)$, and the function $f:[\rho_1,\rho_2]\times R$ ⍰ R is assumed to be smooth in all of its arguments, and the coefficient $\alpha(t)$ is a bounded real-valued function. The parameters $\beta \in R$ are given constants, and $\zeta \in (0,1)$ represents a fractional order, and the scalar $\xi \in (0, T)$ is a fixed interior point. Let $t \in J = [\rho_1,\rho_2]$, where $0 \le \rho_1 < \rho_2$ denotes a fixed interval.

The arrangement of the paper is as follows: Section 2 outlines the essential preliminary concepts and mathematical tools required for the analysis. In Section 3, we establish conditions that ensure the solution is exactly one for the BVP involving Caputo VO-FDE. Section 4 demonstrates a numerical example that validates the uniqueness of the solution. Finally, Section 5 summarizes the key conclusions concerning the uniqueness results obtained in this.

2. Preliminaries

This section presents the essential preliminaries and definitions associated with fractional differential equations of variable order.

Definition 2.1 ([5]) The variable-order Riemann-Liouville fractional integral of a function u with order μ(t) is expressed by

$$I^{\mu(t)}u(t)=\int_{\rho_1}^{t}\frac{(t-\theta)^{\mu(\theta)-1}}{\Gamma(\mu(\theta))}u(\theta)d\theta,\ t>\rho_1\,.$$

Definition 2.2 ([5]) The Caputo variable-order $\mu(t) \in (0,1)$ fractional derivative for a function u is defined as

$${}^{C}D^{\mu(t)}u(t)=\int_{\rho_1}^{t}\frac{(t-\theta)^{-\mu(\theta)}}{\Gamma(1-\mu(\theta))}u'(\theta)d\theta, 0<\mu(t)<1$$

If $\mu(t) = 0$, then ${}^{C}D_t^{\mu(t)}u(t) = u(t)$.

Theorem 2.1 ([4]) Consider a Banach space X and a closed subset $B_{\mathbb{F}} \subset X$. Let the operator $\mathbb{F}: B_{\mathbb{F}} \to B_{\mathbb{F}}$ satisfy the contraction condition

$$\|\mathbb{F}(\hat{u})-\mathbb{F}(u)\|_{\infty}\le L_f\,\|\hat{u}-u\|_{\infty},$$

for $L_f \in (0,1)$ and for all $\hat{u},u\in B_{\mathbb{F}}$. Then, $\mathbb{F}$ has exactly one fixed point $u^{*}\in B_{\mathbb{F}}$.

Assume $f(t,u)$ and $\mu(t)$ satisfy conditions ensuring the BVP is well-posed.

(H1) There is a constant $G_f > 0$ so that $|f(t,u)| \le G_f|u|$ for all $t \in [\rho_1,\rho_2]$, $u \in R$.

(H2) There is a constant $L_f > 0$ so that

$|f(t,u_1) - f(t,u_2)| \le L_f|u_1 - u_2|$ for all $t \in [\rho_1,\rho_2]$.

(H3) Let $\alpha \in C([\rho_1,\rho_2])$, Then $\|\alpha\|_\infty < \infty$.

Lemma 2.1 A function $u \in C([\rho_1,\rho_2])$ is a solution of the BVP (1) if and only if u satisfies the relation

$$u(t)=\beta-\frac{1}{\Gamma(\zeta)}\int_0^{\xi}(\xi-\theta)^{\zeta-1}u(\theta)d\theta+ \int_0^{t}\frac{(t-\theta)^{\mu(\theta)-1}}{\Gamma(\mu(\theta))}(f(\theta,u(\theta))-\alpha(\theta)u(\theta))d\theta$$

for every $t \in [\rho_1,\rho_2]$, where $u_0=\beta-\frac{1}{\Gamma(\zeta)}\int_0^{\xi}(\xi-\theta)^{\zeta-1}u(\theta)d\theta$

3. Uniqueness

In this section, we establish a condition that guarantees exactly one solution for the VO-FDE given in (1). The following theorem provides the main analytical tool for ensuring that the BVP has a single solution.

Let the interval $J = [\rho_1,\rho_2]$ be divided into m consecutive subintervals: $\rho_1 = t_0 < t_1 < t_2 < \cdots < t_m = \rho_2$. Each subinterval is denoted by $J_i = (t_{i-1},t_i]$, $i = 1,2,...,m$.

On each subinterval, define a piecewise constant variable fractional-order function $\mu(t)$ by $\mu(t)=\sum_{i=1}^{m}\mu_i 1_{J_i}(t)$ where $\mu_i \in (0,1]$ is the local fractional order on J_i, and $\mathbf{1}_{Ji}(t)$ is the indicator function: $1_{J_i}(t)=\begin{cases}1, & t\in J_i,\\ 0, & t\notin J_i.\end{cases}$ Assume that $u \in C^1(J_i,R)$ is continuously differentiable.

Then, the Caputo VO-FDE on each subinterval can be written as

$${}^{C}D^{\mu_i}u(t)=\sum_{i=1}^{m}\frac{1}{\Gamma(1-\mu_i)}\int_{J_i\cap[0,T]}(t-\theta)^{-\mu_i}u'(\theta)d\theta$$

Theorem 3.1. Consider the assumptions (H1)-(H3) satisfied. Then, the BVP associated with the Caputo VO-FDE (1) possesses a unique solution whenever $k_i < 1$.

$$k_i=\left[\frac{\xi^{\zeta}}{\Gamma(\zeta+1)}+\frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(L_f+\|\alpha\|_{\infty}\right)\right]$$

Proof. Define the operator $\mathbb{F}: C^1([\rho_1,\rho_2]) \to C^1([\rho_1,\rho_2])$ by

$$(\mathbb{F}u)(t)=\beta-\frac{1}{\Gamma(\zeta)}\int_0^{\xi}(\xi-\theta)^{\zeta-1}u(\theta)d\theta+ \int_0^{t}\frac{(t-\theta)^{\mu_i-1}}{\Gamma(\mu_i)}(f(\theta,u(\theta))-\alpha(\theta)u(\theta))d\theta$$

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Step 1: Boundedness

Let $B_{\mathbb{F}} = \{u \in C^1([\rho_1,\rho_2]) : \|u\|_\infty \le R_{\mathbb{F}}\}$ for some $R_{\mathbb{F}} > 0$. For $u \in B_{\mathbb{F}}$,

$$
\begin{aligned}
|(\mathbb{F}u)(t)| \le \quad & |\beta| + \frac{1}{\Gamma(\zeta)}\int_0^{\xi}(\xi-\theta)^{\zeta-1}|u(\theta)|\,d\theta + \int_0^{\rho_2}\frac{(t-\theta)^{\mu_i-1}}{\Gamma(\mu_i)} \\
& (|f(\theta,u(\theta))| + |\alpha(\theta)||u(\theta)|)d\theta \\
\le & \left[|\beta| + \frac{\xi^\zeta}{\Gamma(\zeta+1)} + \frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(G_f + \|\alpha\|_\infty\right)\right]\|u\|_\infty \\
\le & \left[|\beta| + \frac{\xi^\zeta}{\Gamma(\zeta+1)} + \frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(G_f + \|\alpha\|_\infty\right)\right]R_{\mathbb{F}}
\end{aligned}
$$

Choose an $R_{\mathbb{F}}$ large enough such that.

$$
|\beta| + \frac{R_{\mathbb{F}}\xi^\zeta}{\Gamma(\zeta+1)} + R_{\mathbb{F}}\frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(G_f + \|\alpha\|_\infty\right) \le R_{\mathbb{F}}
$$

$$
|(\mathbb{F}u)(t)| \le \left[|\beta| + \frac{\xi^\zeta}{\Gamma(\zeta+1)} + \frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(G_f + \|\alpha\|_\infty\right)\right]R_{\mathbb{F}} \le R_{\mathbb{F}}
$$

Hence, $(B_{\mathbb{F}}) \subseteq B_{\mathbb{F}}$, showing boundedness.

Step 2: Contraction

Let $u_1, u_2 \in B_{\mathbb{F}}$. Then

$$
\begin{aligned}
\left|(\mathbb{F}u_1)(t) - (\mathbb{F}u_2)(t)\right| \le & \frac{1}{\Gamma(\zeta)}\int_0^{\xi}(\xi-\theta)^{\zeta-1}\left|u_1(\theta)-u_2(\theta)\right|d\theta + \int_0^{\rho_2}\frac{(t-\theta)^{\mu_i-1}}{\Gamma(\mu_i)} \\
& \left(\left|f(\theta,u_1(\theta)) - f(\theta,u_2(\theta))\right| + |\alpha(\theta)|\left|u_1(\theta)-u_2(\theta)\right|\right)d\theta \\
\le & \frac{1}{\Gamma(\zeta)}\int_0^{\xi}(\xi-\theta)^{\zeta-1}\left|u_1(\theta)-u_2(\theta)\right|d\theta \\
& + \left(L_f + \|\alpha\|_\infty\right)\int_0^{\rho_2}\frac{(t-\theta)^{\mu_i-1}}{\Gamma(\mu_i)}\left|u_1(\theta)-u_2(\theta)\right|d\theta \\
\le & \left[\frac{\xi^\zeta}{\tilde{A}(\zeta+1)} + \frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(L_f + \|\alpha\|_\infty\right)\right]u_1 - u_{2\infty}
\end{aligned}
$$

$|(\mathbb{F}u_1)(t) - (\mathbb{F}u_2)(t)| \le k_i\|u_1 - u_2\|_\infty$. $\left|(\mathbb{F}u_1)(t) - (\mathbb{F}u_2)(t)\right| \le k_i u_1 - u_{2\infty}$

If we choose parameters such that $k_i = \left[\frac{\xi^\zeta}{\Gamma(\zeta+1)} + \frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(L_f + \|\alpha\|_\infty\right)\right] < 1$, then

$\mathbb{F}$ is a contraction. By Theorem 2.1, $\mathbb{F}$ has exactly one fixed point $u \in B_{\mathbb{F}}$.

4. Examples

We present illustrative examples to validate the Piecewise Constant Variable Order Fractional BVP discussed in the preceding section.

Consider the BVP on $J = [0,0.5]$:

$$
{}^C D^{\mu(t)}u(t) + \alpha(t)u(t) = f(t,u(t)),\ t \in [0,0.5],
$$

with the condition $u(0) + ({}_0I^{\zeta}u)(0.4) = \beta$, where $\beta = 1$, $\zeta = 0.4$, and the coefficient $\alpha(t) = 0.2t$.

Solution: Divide J into two subintervals:

$J_1 = [0,0.25]$, $J_2 = (0.25,0.5]$.

Define the piecewise constant variable fractional order:

$$
\mu(t) = \begin{cases} 0.4, & t \in J_1, \\ 0.6, & t \in J_2. \end{cases}'
$$

Hence, the problem becomes:

$$
\begin{cases}
{}^C D^{0.6}u(t) + (0.2t)u(t) = 0.1t + 0.06\arctan(u(t)) + 0.04\dfrac{u(t)}{1+u(t)^2},\ t \in (0,0.25] \\
{}^C D^{0.8}u(t) + (0.2t)u(t) = 0.1t + 0.06\arctan(u(t)) + 0.04\dfrac{u(t)}{1+u(t)^2},\ t \in (0.25,0.5] \\
u(0) + \left({}_0I^{0.4}u\right)(0.4) = 1
\end{cases}
$$

Compute the Lipschitz constant with respect to u:

$$
\begin{aligned}
\left|f(t,u_1) - f(t,u_2)\right| \quad & \le 0.06\left|\arctan(u_1) - \arctan(u_2)\right| + 0.04\left|\frac{u_1}{1+u_1^2} - \frac{u_2}{1+u_2^2}\right| \\
& \le (0.06 + 0.04)\left|u_1 - u_2\right| \\
& \le L_f\left|u_1 - u_2\right|
\end{aligned}
$$

Hence, the Lipschitz constant is $L_f = 0.1$.

Uniqueness:

$$
k_i = \frac{\xi^\zeta}{\Gamma(\zeta+1)} + \frac{(\rho_2-\rho_1)^{\mu_i}}{\Gamma(\mu_i+1)}\left(L_f + \|\alpha\|_\infty\right)
$$

Take $\xi = 0.4$, $\rho_2 - \rho_1 = 0.5$, $\|\alpha\|_\infty = 0.1$, $L_f = 0.1$:

$$
k_1 = \frac{0.4^{0.4}}{\Gamma(1.4)} + \frac{0.5^{0.4}}{\Gamma(1.4)}(0.1+0.1) \approx 0.780 + 0.171 \approx 0.951 < 1
$$

$$
k_2 = \frac{0.4^{0.4}}{\Gamma(1.4)} + \frac{0.5^{0.6}}{\Gamma(1.6)}(0.1+0.1) \approx 0.780 + 0.148 \approx 0.928 < 1
$$

Hence, the condition $k_1, k_2 < 1$ is satisfied, guaranteeing the example has exactly one solution (Figures 1-3).

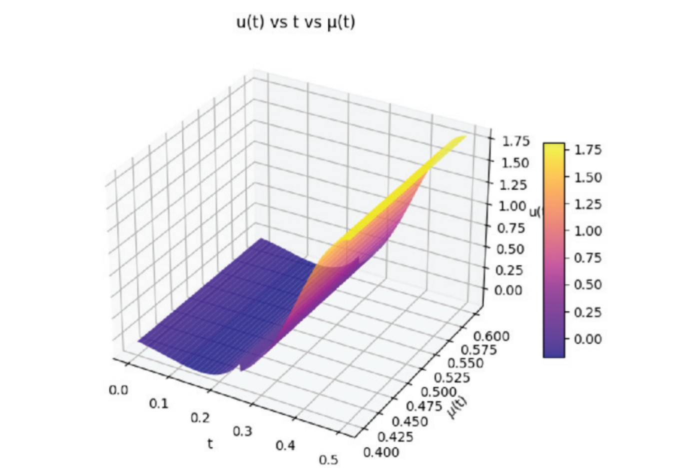

Figure 1: 3D surface showing the numerical solution $u(t)$ over time t with the corresponding variable order $\mu(t)$.

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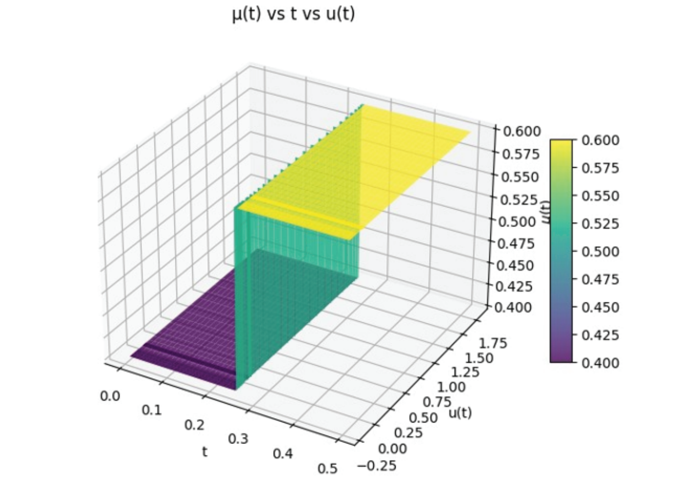

Figure 2: Variation of the fractional order $\mu(t)$ with time t, highlighting the piecewise-constant behavior and its influence on $u(t)$.

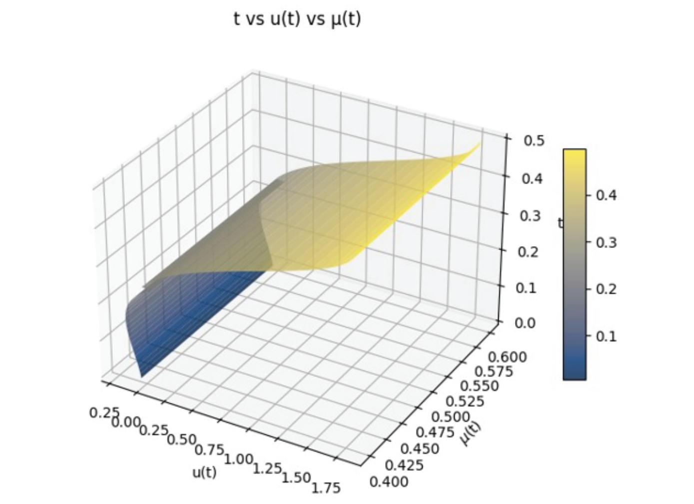

Figure 3: Temporal evolution of t as a function of $u(t)$ and $\mu(t)$, emphasizing the dependency of solution dynamics on the variable order.

The three 3D plots display how $u(t)$ evolves with time and the variable order $\mu(t)$. The solution remains smooth and single valued throughout $[0, T]$, providing numerical confirmation of its uniqueness.

5. Conclusion

This study has established a condition that guarantees the uniqueness of the solution for the BVP involving Caputo VO-FDE. By employing the Banach contraction principle, it is shown that under appropriate assumptions on the nonlinear function and the coefficients, the BVP admits exactly one solution. The theoretical findings are further supported by illustrative examples, confirming the practical validity of the uniqueness result.

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