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Short Communication

Analytic Solving of Equations of Polynomial Type in Variables and Derivatives: A Unified Calculus Based on Power Geometry

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Abstract

Equations that are polynomial in variables and their derivatives appear throughout algebraic geometry, dynamical systems, and the theory of differential equations. This article presents a unified analytic calculus for obtaining asymptotic expansions and simplified representations of solutions to such equations. The approach relies on the systematic use of power geometry, including truncated equations, power transformations, logarithmic transformations, and normalizing coordinate changes. The calculus applies uniformly to algebraic equations, ordinary differential equations, autonomous systems, and partial differential equations. This expanded version provides a structured methodological framework, clarifies the stepwise procedure, and illustrates its relevance through conceptual applications. The methods presented here support the systematic construction of asymptotic solutions and enable the analytical treatment of nonlinear problems that often resist classical techniques.

Introduction

Polynomial-type equations involving variables and their derivatives arise in a wide range of mathematical and physical models. Examples include algebraic varieties near singularities, nonlinear ordinary differential equations (ODEs), autonomous dynamical systems, and certain classes of partial differential equations (PDEs). Analytical methods for solving such equations often rely on special representations, asymptotic expansions, or geometric interpretations of differential orders [1].

To address these challenges, an analytic calculus has been developed that enables the systematic construction of asymptotic solutions. This calculus is based on the concepts of power geometry, truncated equations, and carefully selected coordinate transformations. The central idea is to reduce a complex polynomial-type equation to a hierarchy of simplified equations whose solutions approximate the behavior of the original system.

Purpose of the study

The purpose of this article is to present a structured and unified methodology for solving equations of polynomial type analytically. In particular, we aim to:

- Describe each step of the calculus in a clear procedural manner,
- Explain the role of truncated equations and normalizing transformations,
- Show how power geometry generalizes classical order-based methods,
- Highlight conceptual applications across algebraic, ODE, and PDE settings.

Significance

Traditional tools such as Newton's polyhedron or standard asymptotic methods are often limited when confronted with nonstandard orders of derivatives or nonlinear perturbations [2,3]. Power geometry expands these classical techniques by allowing more general relationships between orders of variables and derivatives, thereby enabling new asymptotic forms [4], including:

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- Power expansions with non-integer exponents,
- Expansions with oscillatory, trigonometric, or elliptic coefficients,
- Solutions with nonstandard derivative order gaps.

The analytic calculus used in this work was developed in earlier studies on nonlinear analysis, truncated equations, parametric expansions, and power geometry [5-14].

Structure of the paper

This paper is organized into four main sections:

- **Section 2** describes the analytic calculus and its methodological foundations.
- **Section 3** presents conceptual applications to various equation types.
- **Section 4** discusses advantages, limitations, and structural implications.
- **Section 5** concludes with a summary of contributions and future directions.

Methods: A unified calculus based on power geometry

The analytic calculus consists of five sequential steps applicable to any polynomial-type equation. These steps are summarized below.

Step 1: Selection of truncated equations

A truncated equation consists of the dominant monomials from the original equation. To determine them, one constructs a *supersupport*—a geometric representation that includes each monomial's exponent vector and coefficient magnitude.

For algebraic equations, these points form an Adamar polyhedron, whose faces determine admissible truncated equations [10]. For differential equations, the geometric structure additionally incorporates derivative orders.

Step 2: Power and logarithmic transformations

Each truncated equation is simplified using [11,12]:

- Power transformations $y = x^{\alpha} z$,
- Logarithmic coordinate changes $x = e^{t}$,

To convert the equation into a form with a simple leading-order solution.

This transformation may be repeated several times until a solvable representation is reached.

Step 3: Construction of a leading-order solution

The simplified truncated equation yields a primary asymptotic form. This may include:

- Polynomial-type solutions,
- Fractional power solutions,
- Trigonometric or elliptic coefficient functions (in higher-level power geometry).

Step 4: Perturbation and normalizing transformation

The leading-order solution is substituted into the full equation. If the resulting perturbation contains a linear part, a normalizing transformation is applied to construct the full asymptotic expansion.

If no linear part exists, a new truncated equation is extracted and the process iterates.

Step 5: Higher-level power geometry

Classical power geometry assumes

ord $(y') = \text{ord}(y) - 1$.

This is the *zero level.*

Higher-level power geometry removes this restriction and allows arbitrary differences between the orders of successive derivatives. As a result, one may obtain expansions whose coefficients are:

- Trigonometric functions,
- Elliptic functions,
- Periodic or quasiperiodic functions.

This generalization produces entirely new families of solutions inaccessible to classical methods [7,8,13,14].

Applications to different classes of equations

Algebraic equations

For an algebraic equation

$f(X) = \sum a_Q X^Q = 0$

Each monomial defines a point in the supersupport space.

Truncated equations derived from the faces of the Adamar polyhedron allow the study of singularities that Newton's polyhedron cannot resolve [6].

This enables the derivation of:

- Parametric expansions near singular points,
- Asymptotic representations of algebraic varieties,
- Multivariable solutions.

Ordinary differential equations

For a single ODE or an autonomous system, the calculus provides:

- Power expansions near equilibrium points,

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- Expansions with oscillatory coefficients (using higher-level techniques),
- Asymptotic descriptions in multidimensional systems [8,11,12].

Partial differential equations

Although less developed in current literature, the same approach can be extended conceptually to PDEs [9,15]:

- Defining generalized supersupports in higher-dimensional order spaces,
- Applying power geometry to mixed partial derivatives,
- Constructing multi-parameter expansions.

This represents a promising direction for future research.

Discussion

That distinguish it from traditional asymptotic and symmetry-based methods [1-4]:

1. **Generality:** applicable to algebraic, ODE, system, and PDE settings.
2. **Flexibility:** permits nonstandard orders and nonclassical solution forms.
3. **Constructiveness:** provides explicit algorithms and transformations.
4. **Extensibility:** higher-level power geometry offers new asymptotic structures.

Limitations include:

- The need for computational support for complex transformations,
- Increased geometric complexity for PDEs,
- Sensitivity to the choice of appropriate truncated equations.

Recent software developments, however, greatly support automation of these procedures.

Conclusion

This article presents a structured and expanded exposition of an analytic calculus for solving polynomial-type equations involving variables and derivatives. Through power geometry, truncated equations, and coordinate transformations, the methodology unifies and extends classical asymptotic techniques. The framework is broadly applicable across algebraic, differential, and dynamical systems and provides powerful tools for investigating nonlinear phenomena.

Future work includes formalizing supersupport structures for PDEs, automating transformation sequences, and integrating these methods with symbolic computation systems.

Declarations

Author contributions: The author conceptualized the study, developed the methodology, performed the analysis, and wrote the manuscript.

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