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Review Article

Some Hermite–Hadamard-mercer Inequalities on the Coordinates on Post Quantum

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Abstract

In this paper, we develop new Hermite-Hadamard-Mercer type inequalities on coordinates via post-quantum calculus, also known as (p, q) - calculus. By introducing novel (p_1, p_2, q_1, q_2)-differentiable and (p_1, p_2, q_1, q_2)-integrable functions, we generalize classical results and extend previous inequalities under the setting of coordinate convexity. Several new identities are derived, which naturally reduce to known results when specific parameters are chosen. Numerical examples and visualizations are also provided to illustrate the utility of our results.

1. Introduction

Mathematical inequalities are fundamental in both pure and applied mathematics, offering essential tools in areas ranging from analysis to physics. Among these, the Hermite-Hadamard inequality for convex functions plays a pivotal role, which is defined as follows:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

The Hermite-Hadamard inequality and a variety of refinements of Hermite-Hadamard inequality have been extensively studied by many researchers, including those involving Jensen and Mercer-type refinements.

Jensen inequality has been caught attention of many researshers, and many articles related to different versions of this inequality have been found in the literature. Jensen's inequality can be given as follows:

Let $0 < x_1 \le x_2 \le \ldots \le x_n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be non-negative weights such that $\sum_{k-1}^{n} u_k = 1$ if f is convex function on the interval $[a, b]$, then

$$f\left(\sum_{k=1}^{n} u_k x_k\right) \le \sum_{k=1}^{n} u_k f(x_k),$$

where every $x_k \in [a,b]$ and all $\mu_k \in [0,1]$.

A new variant of Jensen inequality thast has been established by Mercer can be presented as follows:

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In 2003, Mercer [1] proved another version of Jensen inequality, which is called Jensen-Mercer inequality and stated as follows.

Theorem 1.1

For a convex mapping $f:[a,b]\to\mathbb{R}$, for following inequality holds for each $x_j\in[a,b]$:

$$f\left(a+b-\sum_{j=1}^{n}u_jx_j\right)\le f(a)+f(b)-\sum_{j=1}^{n}u_j f(x_j),$$

where $u_j\in[0,1]$ and $\sum_{j=1}^{n}u_j=11$.

In 2013, Kian, et al. [2] used this new Jensen inequality and established the following new versions of Hermite-Hadamard inequality:

Theorem 1.2

For a convex mapping $f:[a,b]\to\mathbb{R}$, for following inequality holds for each $x,y\in[a,b]$ and $x<y$:

$$f\left(a+b-\frac{x+y}{2}\right)\le f(a)+f(b)-\frac{1}{y-x}\int_x^y f(u)du\le f(a)+f(b)-f\left(\frac{x+y}{2}\right)$$

and

$$f\left(a+b-\frac{x+y}{2}\right)\le\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(u)du$$

$$\le\frac{f(a+b-x)+f(a+b-y)}{2}$$

$$\le f(a)+f(b)-f\left(\frac{x+y}{2}\right).$$

The ordinary calculus of Newton and Leibniz is well known to be investigated extensively and intensively to produce a large number of related formulas and properties as well as applications in a variety of fields ranging from natural sciences to social sciences.

Recent studies have explored these inequalities using quantum and post-quantum calculus, which extend traditional calculus by means of q and (p,q)-analogues.

Quantum calculus, which is often known as q-calculus or calculus without limits, is based on finite difference. In quantum calculus we obtain q-analogues of mathematical objects which can be recaptured by taking. The history of q-calculus can be track to Euler, who first introduced q-calculus in the track of Newton's work on infinite series.

Then, in 1910, F. H. Jackson presented a systematic study of q-calculus and defined the q-defined integral, which is known as the q-Jackson integral. In recent years, the interest in q-calculus been arising due to high demand of mathematics in this field. The q-calculus numerous applications in various fields of mathematics and other areas such as combinatory, dynamical systems, fractals, number theory, orthogonal polynomials, special functions, mechanics and also for scientific problems in some applied areas.

In 2013, Tariboon and Ntouyas defined new q-derivatives and q-integrals of a continuous function on a finite interval. These definitions have been studied in various inequalities, for example, Hermiter-Hadamard inequalities, Ostrowski inequalities, Fejér inequalities, Simpson inequalities and Newton inequalities, and the references cited therein [3-12].

Along with the development of the theory and application of q-calculus, the theory of q-calculus based on two parameters (p-q)-integral has also presented and received more attention during the last few decades.

Recent, Ali, et al. [3] and Sitthiwirattham, et al. [13] ued new techniques to prove the following two different and new versions of Hermite-Hadamard type inequalitites:

Theorem 1.3

For a convex mapping $f:[a,b]\to\mathbb{R}$, for following inequality holds:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}f(x)^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^{b}f(x)_{\frac{a+b}{2}}d_qx\right]\le\frac{f(a)+f(b)}{2},$$

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$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\left[\int_{a}^{\frac{a+b}{2}}f(x)_{a}\,d_{q}x+\int_{\frac{a+b}{2}}^{b}f(x)^{b}\,d_{q}x\right]\leq\frac{f(a)+f(b)}{2}.$$

Remark

By setting the limit as q 1^- in above Theorem, we recapture the traditional Hermite-Hadamard inequality.

Post-quantum calculus, also called (p,q)-calculus, is another generalization of q-calculus on the interval . The (p,q)-calculus consists of two-parameter quantum calculus (p and q-numbers) which are independent. The (p,q)- calculus was first introduced by Chakrabarti and Jagannathan in 1991. Then, the new (p,q)-deravative and (p,q)-integral of a continuous function on finite interval were by Tunc and Gov in 2016. In (p,q)-calculus, we obtain q-calculus formula for case p=1, and then get classical formula for case of . Base on (p,q)-calculus, many literatures have been published by many researchers, see [14-25] for more details and the references cited therein.

In this paper, we continue in this direction by developing Hermite-Hadamard-Mercer type inequalities in the framework of post-quantum calculus on coordinates. Our main contributions include novel identities for functions of two variables involving mixed partial $(p_1; p_2; q_1; q_2)$ - derivatives and integrals.

2. Notation and preliminaries

The following is the brief introduction of the research of post-quantum calculus. Throughout this topic, we let $p_1; p_2; q_1; q_2$ be constants with $0<q_1<p_1\leq 1$ and $0<q_2<p_2\leq 1$ with $[a,b]\subseteq\mathbb{R}$.

Then, for any real number, the (p_1, q_1) - analogue and (p_2, q_2) - analogue of m,n is defined by

$$[m]_{p_1,q_1}=\frac{p_1^m-q_1^m}{p_1-q_1}=p_1^{m-1}+p_1^{m-2}q_1+\ldots+p_1q_1^{m-2}+q_1^{m-1}$$

and

$$[n]_{p_2,q_2}=\frac{p_2^n-q_2^n}{p2-q_2}=p_2^{n-1}+p_2^{n-2}q_2+\ldots+p_2q_2^{n-2}+q_2^{n-1},$$

which is generalization of the $q1$- analogue such that

$$[m]_{q_1}=\frac{1-q_1^m}{1-q_1}=1+q_1+\ldots+q_1^{m-2}+q_1^{m-1}$$

and

$$[n]_{q_2}=\frac{1-q_2^n}{1-q_2}=1+q_2+\ldots+q_2^{n-2}+q_2^{n-1}.$$

Definition 2.1 [26]

If $f:[a,b]\to\mathbb{R}$ is a continuous function, then (p, q) - derivative of the function on $[a, b]$ by

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{p-q},\quad x\neq 0$$

with $0<q<p\leq 1$.

Definition 2.2 [27]

If $f:[a,b]\to\mathbb{R}$ is a continuous function, then $(p, q)_a$ - derivative of the function at x is defined by

$${}_{a}D_{p,q}f(x)=\frac{f(px+(1-p)a)-f(qx+(1-q)a)}{(p-q)(x-a)},\quad x\neq a$$

with $0<q<p\leq 1$.

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For $x = a$, we state ${}_aD_{p,q}f(a)=\lim\limits_{x\to a}{}_aD_{p,q}f(x)$ if it exists and it is finite.

Definiiton 2.3 [16]

If $f:[a,b]\to\mathbb{R}$ is a continuous function, then $(p, q)^b$ – derivative of the function at x is defined by

$$ {}^bD_{p,q}f(x)=\frac{f(qx+(1-q)b)-f(px+(1-p)b)}{(p-q)(b-x)}, \quad x\neq b $$

with $0<q<p\leq 1$.

For $x = b$, we state ${}^bD_{p,q}f(a)=\lim\limits_{x\to b}{}^bD_{p,q}f(x)$ if it exists and it is finite.

Definition 2.4 [26]

If $f:[a,b]\to\mathbb{R}$ is a continuous function and $0<a<b$, then the (p, q) – integral is defined by

$$ \int_a^b f(x)d_{p,q}x=(p-q)(b-a)\sum_{k=0}^{\infty}f\left(\frac{q^k}{p^{k+1}}b+\left(1-\frac{q^k}{p^{k+1}}\right)a\right) $$

with $0<q<p\leq 1$.

Definition 2.5 [27]

If $f:[a,b]\to\mathbb{R}$ is a continuous function and $0<a<b$, then the $(p, q)_a$ – integral is defined by

$$ \int_a^x f(x)_a\, d_{p,q}x=(p-q)(x-a)\sum_{k=0}^{\infty}f\left(\frac{q^k}{p^{k+1}}x+\left(1-\frac{q^k}{p^{k+1}}\right)a\right) $$

with $0<q<p\leq 1$.

Definition 2.6 [16]

If $f:[a,b]\to\mathbb{R}$ is a continuous function and $0<a<b$, then the $(p, q)^b$ – integral is defined by

$$ \int_x^b f(x)^b\, d_{p,q}x=(p-q)(b-x)\sum_{k=0}^{\infty}f\left(\frac{q^k}{p^{k+1}}b+\left(1-\frac{q^k}{p^{k+1}}\right)x\right) $$

with $0<q<p\leq 1$.

In [14], Ali et al. established the Hermite-Hadamard type inequlities on post quantum calculus.

Theorem 2.7

If $f:[a,b]\to\mathbb{R}$ is a convex differentiable function on $[a,b]$, then the $(p, q)^b$ – Hermite-Hadamard inequalities are as follows:

$$ f\left(\frac{pa+qb}{p+q}\right)\leq\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}f(x)^b\, d_{p,q}x\leq\frac{qf(a)+pf(b)}{p+q}. $$

In [27], Tunc and Gov extend the Holdër inequalies ion post-quantum calculus.

Theorem 2.8

If $f:[a,b]\to\mathbb{R}$ is a continuous function and r, s > 0 with $\frac{1}{r}+\frac{1}{s}=1$, then

$$ \int_a^b |f(x)g(x)|_a\, d_{p,q}x\leq\left(\int_a^b |f(x)|^r_a\, d_{p,q}x\right)^{\frac{1}{r}}\left(\int_a^b |g(x)|^s_a\, d_{p,q}x\right)^{\frac{1}{s}}. $$

In [28], H. Kalsoom, et al. introduced the following notions of post-quantum partial derivatives:

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Definition 2.9

Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is a continuous function of two variables. Then the derivatives are given by

$$\frac{{}_{a,c}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y}=\frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(y-c)}\times$$
$$\Big[f\big(q_1x+(1-q_1)a,q_2y+(1-q_2)c\big)$$
$$-f\big(q_1x+(1-q_1)a,p_2y+(1-p_2)c\big)$$
$$-f\big(p_1x+(1-p_1)a,q_2y+(1-q_2)c\big)$$
$$+f\big(p_1x+(1-p_1)a,p_2y+(1-p_2)c\big)\Big],$$

for $x\neq a,\ y\neq c$.

$$\frac{{}_{c}^{b}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y}=\frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(y-c)}\times$$
$$\Big[f\big(q_1x+(1-q_1)b,p_2y+(1-p_2)c\big)$$
$$-f\big(p_1x+(1-p_1)b,p_2y+(1-p_2)c\big)$$
$$-f\big(q_1x+(1-q_1)b,q_2y+(1-q_2)c\big)$$
$$+f\big(p_1x+(1-p_1)a,q_2y+(1-q_2)c\big)\Big],$$

for $x\neq b,\ y\neq c$.

$$\frac{{}_{a}^{d}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1}x\,{}^{d}\partial_{p_2,q_2}y}=\frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(d-y)}\times$$
$$\Big[f\big(p_1x+(1-p_1)a,q_2y+(1-q_2)d\big)$$
$$-f\big(q_1x+(1-q_1)a,q_2y+(1-q_2)d\big)$$
$$-f\big(p_1x+(1-p_1)a,p_2y+(1-p_2)d\big)$$
$$+f\big(q_1x+(1-q_1)a,p_2y+(1-p_2)d\big)\Big],$$

for $x\neq a,\ y\neq d$,

and

$$\frac{{}^{b,d}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1}x\,{}^{d}\partial_{p_2,q_2}y}=\frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(d-y)}\times$$
$$\Big[f\big(q_1x+(1-q_1)b,q_2y+(1-q_2)d\big)$$
$$-f\big(p_1x+(1-p_1)b,q_2y+(1-q_2)d\big)$$
$$-f\big(q_1x+(1-q_1)b,p_2y+(1-p_2)d\big)$$
$$+f\big(p_1x+(1-p_1)b,p_2y+(1-p_2)d\big)\Big],$$

for $x\neq b,\ y\neq d$.

Definition 2.10 [28]

Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is a continuous function of two variables. Then the definite (p_1, p_2, q_1, q_2) – integral are given by

$$\int_a^x\int_c^y f(t,s)\,{}_{c}d_{p_2,q_2}s\,{}_{a}d_{p_1,q_1}t$$
$$=(p_1-q_1)(p_2-q_2)(x-a)(y-c)$$

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$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right),$$

for $(x,y)\in[a,b]\times[c,d]$.

Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is a continuous function of two variables. Then the definite (p_1, p_2, q_1, q_2) – integral are given by

$$\int_x^b\int_c^y f(t,s)_c\,d_{p_2,q_2}s^b d_{p_1,q_1}t$$

$$=(p_1-q_1)(p_2-q_2)(b-x)(y-c)$$

$$\times\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)b,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right),$$

for $(x,y)\in[a,b]\times[c,d]$.

Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is a continuous function of two variables. Then the definite (p_1, p_2, q_1, q_2) – integral are given by

$$\int_a^x\int_y^d f(t,s)_c\,d_{p_2,q_2}s_a d_{p_1,q_1}t$$

$$=(p_1-q_1)(p_2-q_2)(x-a)(d-y)$$

$$\times\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right),$$

for $(x,y)\in[a,b]\times[c,d]$

and

Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is a continuous function of two variables. Then the definite (p_1, p_2, q_1, q_2) –integral are given by

$$\int_x^b\int_y^d f(t,s)_c\,d_{p_2,q_2}s_a d_{p_1,q_1}t$$

$$=(p_1-q_1)(p_2-q_2)(b-x)(d-y)$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)b,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right),$$

for $(x,y)\in[a,b]\times[c,d]$.

Lemma 2.11 [25] (p_1, p_2, q_1, q_2) – Hölder's inequality for functions of two variables

Let f, g be (p_1, p_2, q_1, q_2) – integrable functions on $[a,b]\times[c,d]$ and $\frac{1}{r}+\frac{1}{s}=1$ with $r, s > 1$. Then, we have

$$\int_a^x\int_c^y |f(t,s)g(t,s)|_c\,d_{p_2,q_2}s_a d_{p_1,q_1}t$$

$$\le\left(\int_a^x\int_c^y |f(t,s)|^r{}_c\,d_{p_2,q_2}s_a d_{p_1,q_1}t\right)^{\frac{1}{r}}$$

$$\times\left(\int_a^x\int_c^y |g(t,s)|^s{}_c\,d_{p_2,q_2}s_a d_{p_1,q_1}t\right)^{\frac{1}{s}}$$

for all $(x,y)\in[a,b]\times[c,d]$.

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Lemma 2.12 [25] (p_1, p_2, q_1, q_2) – power mean inequality for functions of two variables

Let f, g be (p_1, p_2, q_1, q_2) – integrable functions on $[a,b]\times[c,d]$ and $\alpha \geq 1$. Then, we have

$$\int_a^x\int_c^y |f(t,s)g(t,s)|_c\, d_{p_2,q_2}s_a d_{p_1,q_1}t$$

$$\leq\left(\int_a^x\int_c^y |f(t,s)|_c\, d_{p_2,q_2}s_a d_{p_1,q_1}t\right)^{1-\frac{1}{\alpha}}$$

$$\leq\left(\int_a^x\int_c^y |g(t,s)|^{\alpha}{}_c\, d_{p_2,q_2}s_a d_{p_1,q_1}t\right)^{\frac{1}{\alpha}}$$

for all $(x,y)\in[a,b]\times[c,d]$.

3. Main results

Lemma 3.1

Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2) – differentiable function on $(a,b)\times(c,d)$. If

$\frac{{}_{a,c}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_a\partial_{p_1,q_1}x\ {}_c\partial_{p_2,q_2}y}, \frac{{}^b_c\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^b\partial_{p_1,q_1}x\ {}_c\partial_{p_2,q_2}y}, \frac{{}^d_a\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_a\partial_{p_1,q_1}x^d\partial_{p_2,q_2}y}, \frac{{}^{b,d}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^b\partial_{p_1,q_1}x^d\partial_{p_2,q_2}y}$ are continuous and (p_1, p_2, q_1, q_2) –integrable on $[a,b]\times[c,d]$.

Then we have

$$\frac{1}{p_1p_2q_1q_2(n-m)^2(l-k)^2}$$

$$\times\left[\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t\right.$$

$$+\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$+\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$\left.\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t\right]$$

$$-\frac{1}{p_2(n-m)(l-k)^2}\left[\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(a+b-m,s)d_{p_2,q_2}s\right.$$

$$+\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(a+b-n,s)d_{p_2,q_2}s$$

$$+\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(a+b-m,s)d_{p_2,q_2}s$$

$$\left.\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(a+b-n,s)d_{p_2,q_2}s\right]$$

$$-\frac{1}{p_1(n-m)^2(l-k)}\left[\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]} f(t,c+d-k)d_{p_2,q_2}sd_{p_1,q_1}t\right.$$

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$$+\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m} f(t,c+d-k)\,d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$+\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]} f(t,c+d-l)\,d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$\left.\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m} f(t,c+d-l)\,d_{p_2,q_2}s\,d_{p_1,q_1}t\right]$$

$$+\frac{1}{(n-m)(l-k)}\left[(a+b-m,c+d-k)+(a+b-n,c+d-k)\right.$$

$$\left.+(a+b-m,c+d-l)+(a+b-n,c+d-l)\right]$$

$$={}^{b,d}_{a,c}J_{p_1,p_2,q_1,q_2}f(t,s)$$

where

$${}^{b,d}_{a,c}J_{p_1,p_2,q_1,q_2}f(t,s)$$

$$\int_0^1\int_0^1 ts\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a+b-[tm+(1-t)n],c+d-[sk+(1-s)l]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$\int_0^1\int_0^1 ts\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a+b-[(1-t)m+tn],c+d-[sk+(1-s)l]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$\int_0^1\int_0^1 ts\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f\left(a+b-[tm+(1-t)n],c+d-[(1-s)k+sl]\right)d_{p_2,q_2}s\,d_{p_1,q_1}tt$$

$$\int_0^1\int_0^1 ts\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f\left(a+b-[(1-t)m+tn],c+d-[(1-s)k+sl]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

for $m,n\in[a,b],k,l\in[c,d]$ and $m<n,k<l$.

Proof:

By the definition 2.9, we have

$$\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a+b-[tm+(1-t)n],c+d-[sk+(1-s)l]\right)$$

$$=\frac{1}{(p_1-q_1)(p_2-q_2)t(n-m)s(l-k)}\times$$

$$\left[f\left(a+b-[(1-q_1t)n+q_1tm,c+d-[(1-q_2s)l+q_2sk]]\right)\right.$$

$$-f\left(a+b-[(1-q_1t)n+q_1tm,c+d-[(1-p_2s)l+p_2sk]]\right)$$

$$-f\left(a+b-[(1-p_1t)n+p_1tm,c+d-[(1-q_2s)l+q_2sk]]\right)$$

$$\left.+f\left(a+b-[(1-p_1t)n+p_1tm,c+d-[(1-p_2s)l+p_2sk]]\right)\right].$$

Moreover,

$$\int_0^1\int_0^1 ts\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a+b-[tm+(1-t)n],c+d-[sk+(1-s)l]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\int_0^1\int_0^1 ts\frac{1}{(p_1-q_1)(p_2-q_2)t(n-m)s(l-k)}\times$$

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$$\left[f\left(a+b-\left[(1-q_1t)n+q_1tm\right],c+d-\left[(1-q_2s)l+q_2sk\right]\right)\right.$$

$$-f\left(a+b-\left[(1-q_1t)n+q_1tm\right],c+d-\left[(1-p_2s)l+p_2sk\right]\right)$$

$$-f\left(a+b-\left[(1-p_1t)n+p_1tm\right],c+d-\left[(1-q_2s)l+q_2sk\right]\right)$$

$$\left.+f\left(a+b-\left[(1-p_1t)n+p_1tm\right],c+d-\left[(1-p_2s)l+p_2sk\right]\right)\right]d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\frac{1}{(p_1-q_1)(p_2-q_2)(n-m)(l-k)}\int_0^1\int_0^1$$
$$\left[f\left(a+b-\left[(1-q_1t)n+q_1tm\right],c+d-\left[(1-q_2s)l+q_2sk\right]\right)\right.$$
$$-f\left(a+b-\left[(1-q_1t)n+q_1tm\right],c+d-\left[(1-p_2s)l+p_2sk\right]\right)$$
$$-f\left(a+b-\left[(1-p_1t)n+p_1tm\right],c+d-\left[(1-q_2s)l+q_2sk\right]\right)$$
$$\left.+f\left(a+b-\left[(1-p_1t)n+p_1tm\right],c+d-\left[(1-p_2s)l+p_2sk\right]\right)\right]d_{p_2,q_2}s\,d_{p_1,q_1}t$$
$$=\frac{1}{(p_1-q_1)(p_2-q_2)(n-m)(l-k)}\left[I_1-I_2-I_3+I_4\right].$$

By definition 2.10, we have

$$I_1=\int_0^1\int_0^1 f\left(a+b-\left[(1-q_1t)n+q_1tm,\right]c+d-\left[(1-q_2s)l+q_2sk\right]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)n+\frac{q_1^{n+1}}{p_1^{n+1}}m\right],c+d-\left[\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)l+\frac{q_2^{m+1}}{p_2^{m+1}}k\right]\right)$$

$$=\frac{p_1p_2}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)n+\frac{q_1^n}{p_1^{n+1}}p_1m\right],c+d-\left[\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)l+\frac{q_2^m}{p_2^{m+1}}p_2k\right]\right)$$

$$-\frac{p_2}{q_1q_2}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)n+\frac{q_1^n}{p_1^{n+1}}p_1m\right],c+d-k\right)$$

$$-\frac{p_1}{q_1q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-m,c+d-\left[\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)l+\frac{q_2^m}{p_2^{m+1}}p_2k\right]\right)$$

$$+\frac{1}{q_1q_2}(a+b-m,c+d-k).$$

$$I_2=\int_0^1\int_0^1 f\left(a+b-\left[(1-q_1t)n+q_1tm\right],c+d-\left[(1-p_2s)l+p_2sk\right]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)n+\frac{q_1^{n+1}}{p_1^{n+1}}m\right],c+d-\left[\left(1-\frac{q_2^m}{p_2^m}\right)l+\frac{q_2^m}{p_2^m}k\right]\right)$$

$$=\frac{p_1}{q_1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)n+\frac{q_1^n}{p_1^{n+1}}p_1m\right],c+d-\left[\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)l+\frac{q_2^m}{p_2^{m+1}}p_2k\right]\right)$$

$$-\frac{1}{q_1}\sum_{m=0}^{\infty}\check{z}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-m,c+d-\left[\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)l+\frac{q_2^m}{p_2^{m+1}}p_2k\right]\right).$$

$$I_3=\int_0^1\int_0^1 f\left(a+b-\left[(1-p_1t)n+p_1tm\right],c+d-\left[(1-q_2s)l+q_2sk\right]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

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$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^n}\right)n+\frac{q_1^n}{p_1^n}m\right],c+d-\left[\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)l+\frac{q_2^{m+1}}{p_2^{m+1}}k\right]\right)$$

$$=\frac{p_2}{q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)n+\frac{q_1^n}{p_1^{n+1}}p_1m\right],c+d-\left[\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)l+\frac{q_2^m}{p_2^{m+1}}p_2k\right]\right)$$

$$-\frac{1}{q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)n+\frac{q_1^n}{p_1^{n+1}}p_1m\right],c+d-k\right).$$

$$I_4=\int_0^1\int_0^1 f\left(a+b-\left[(1-p_1t)n+p_1tm\right],c+d-\left[(1-p_2s)l+p_2sk\right]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)n+\frac{q_1^n}{p_1^{n+1}}p_1m\right],c+d-\left[\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)l+\frac{q_2^m}{p_2^{m+1}}p_2k\right]\right)$$

So that,

$$\int_0^1\int_0^1 ts\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a+b-\left[tm+(1-t)n\right],c+d-\left[sk+(1-s)l\right]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\frac{1}{(p_1-q_1)(p_2-q_2)(n-m)(l-k)}\times$$

$$\left[\frac{(p_1-q_1)(p_2-q_2)}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^n}\right)n+\frac{q_1^n}{p_1^n}m\right],c+d-\left[\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)l+\frac{q_2^{m+1}}{p_2^{m+1}}k\right]\right)\right.$$

$$-\frac{(p_2-q_2)}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-m,c+d-\left[\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)l+\frac{q_2^m}{p_2^{m+1}}p_2k\right]\right)$$

$$-\frac{(p_1-q_1)}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(a+b-\left[\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)n+\frac{q_1^n}{p_1^{n+1}}p_1m\right],c+d-k\right)$$

$$+\frac{1}{q_1q_2}(a+b-m,c+d-k)$$

$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(n-m)^2(l-k)^2}\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]}f(t,s)d_{p_2,q_2}s\,d_{p_1,q_1}t\right.$$

$$-\frac{1}{p_2(n-m)(l-k)^2}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]}f(a+b-m,s)d_{p_2,q_2}s$$

$$-\frac{1}{p_1(n-m)^2(l-k)}\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]}f(t,c+d-k)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$\left.+\frac{1}{(n-m)(l-k)}(a+b-m,c+d-k)\right].$$

Similarly, by the equality, we obtain the identities

$$\int_0^1\int_0^1 ts\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a+b-\left[(1-t)m+tn\right],c+d-\left[sk+(1-s)l\right]\right)d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(n-m)^2(l-k)^2}\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]}f(t,s)d_{p_2,q_2}s\,d_{p_1,q_1}t\right.$$

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$$-\frac{1}{p_2(n-m)(l-k)^2}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(a+b-n,s)d_{p_2,q_2}s$$

$$-\frac{1}{p_1(n-m)^2(l-k)}\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m} f(t,c+d-k)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$\left.+\frac{1}{(n-m)(l-k)}(a+b-n,c+d-k)\right].$$

$$\int_0^1\int_0^1 ts\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f\left(a+b-[tm+(1-t)n],c+d-[(1-s)k+sl]\right)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(n-m)^2(l-k)^2}\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t\right.$$

$$-\frac{1}{p_2(n-m)(l-k)^2}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(a+b-m,s)d_{p_2,q_2}s$$

$$-\frac{1}{p_1(n-m)^2(l-k)}\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]} f(t,,c+d-l)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$\left.+\frac{1}{(n-m)(l-k)}(a+b-m,c+d-l)\right].$$

$$\int_0^1\int_0^1 ts\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f\left(a+b-[(1-t)m+tn],c+d-[(1-s)k+sl]\right)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(n-m)^2(l-k)^2}\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t\right.$$

$$-\frac{1}{p_2(n-m)(l-k)^2}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(a+b-n,s)d_{p_2,q_2}s$$

$$-\frac{1}{p_1(n-m)^2(l-k)}\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m} f(t,c+d-l)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$\left.+\frac{1}{(n-m)(l-k)}(a+b-n,c+d-l)\right].$$

Thus, we have

$$\frac{1}{p_1p_2q_1q_2(n-m)^2(l-k)^2}\left[\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t\right.$$

$$+\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m}\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$+\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$\left.\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m}\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(t,s)d_{p_2,q_2}sd_{p_1,q_1}t\right]$$

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$$-\frac{1}{p_2(n-m)(l-k)^2}\left[\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(a+b-m,s)d_{p_2,q_2}s\right.$$

$$+\int_{c+d-l}^{c+d-[(1-p_2)l+p_2k]} f(a+b-n,s)d_{p_2,q_2}s$$

$$+\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(a+b-m,s)d_{p_2,q_2}s$$

$$\left.\int_{c+d-[(1-p_2)k+p_2l]}^{c+d-k} f(a+b-n,s)d_{p_2,q_2}s\right]$$

$$-\frac{1}{p_1(n-m)^2(l-k)}\left[\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]} f(t,c+d-k)d_{p_2,q_2}sd_{p_1,q_1}t\right.$$

$$+\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m} f(t,c+d-k)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$+\int_{a+b-n}^{a+b-[(1-p_1)n+p_1m]} f(t,c+d-l)d_{p_2,q_2}sd_{p_1,q_1}t$$

$$\left.\int_{a+b-[(1-p_1)m+p_1n]}^{a+b-m} f(t,c+d-l)d_{p_2,q_2}sd_{p_1,q_1}t\right]$$

$$+\frac{1}{(n-m)(l-k)}\big[(a+b-m,c+d-k)+(a+b-n,c+d-k)$$

$$+(a+b-m,c+d-l)+(a+b-n,c+d-l)\big]$$

$$={}_{a,c}^{b,d}J_{p_1,p_2,q_1,q_2}f(t,s).$$

Corollary 3.2

If we set p_1, p_2 = 1, then the Lemma 3.1 reduces to the following equality.

$$\frac{4}{q_1q_2(n-m)^2(l-k)^2}\int_{a+b-n}^{a+b-m}\int_{c+d-l}^{c+d-k} f(t,s)d_{q_2}sd_{q_1}t$$

$$-\frac{1}{(n-m)(l-k)^2}\left[2\int_{c+d-l}^{c+d-k} f(a+b-m,s)d_{q_2}s\right.$$

$$\left.+2\int_{c+d-l}^{c+d-k} f(a+b-n,s)d_{q_2}s\right]$$

$$-\frac{1}{(n-m)^2(l-k)}\left[2\int_{a+b-n}^{a+b-m} f(t,c+d-k)d_{q_2}sd_{q_1}t\right.$$

$$\left.+2\int_{a+b-n}^{a+b-m} f(t,c+d-l)d_{q_2}sd_{q_1}t\right]$$

$$+\frac{1}{(n-m)(l-k)}\big[(a+b-m,c+d-k)+(a+b-n,c+d-k)$$

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$$+\left(a+b-m,c+d-l\right)+\left(a+b-n,c+d-l\right)\Big]$$

$$= {}_{a,c}^{b,d}J_{q_1,q_2}f\left(t,s\right)$$

where

$${}_{a,c}^{b,d}J_{q_1,q_2}f\left(t,s\right)$$

$$=\int_0^1\int_0^1 ts\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\ {}_{c+d-l}\partial_{q_2}s}f\left(a+b-\left[tm+(1-t)n\right],c+d-\left[sk+(1-s)l\right]\right)d_{q_2}sd_{q_1}t$$

$$+\int_0^1\int_0^1 ts\frac{{}^{a+b-m}_{c+d-l}\partial_{q_1,q_2}}{{}_{a+b-m}\partial_{q_1}t\ {}_{c+d-l}\partial_{q_2}s}f\left(a+b-\left[(1-t)m+tn\right],c+d-\left[sk+(1-s)l\right]\right)d_{q_2}sd_{q_1}t$$

$$+\int_0^1\int_0^1 ts\frac{{}^{c+d-k}_{a+b-n}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t^{c+d-k}\partial_{q_2}s}f\left(a+b-\left[tm+(1-t)n\right],c+d-\left[(1-s)k+sl\right]\right)d_{q_2}sd_{q_1}t$$

$$+\int_0^1\int_0^1 ts\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{q_1}t^{c+d-k}\partial_{q_2}s}f\left(a+b-\left[(1-t)m+tn\right],c+d-\left[(1-s)k+sl\right]\right)d_{q_2}sd_{q_1}t$$

Theorem 3.3

If the conditions of Lemma 3.1 hold and

$$\frac{{}_{a,c}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}_{a}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y},\frac{{}^{b}_{c}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}^{b}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y},\frac{{}^{d}_{a}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}_{a}\partial_{p_1,q_1}x^{d}\partial_{p_2,q_2}y},\frac{{}^{b,d}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}^{b}\partial_{p_1,q_1}x^{d}\partial_{p_2,q_2}y}$$

are coordinated convex, then we have the following inequality:

$$\left|{}_{a,c}^{b,d}J_{p_1,p_2,q_1,q_2}f\left(t,s\right)\right|$$

$$\leq\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a,c\right)\right|+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a,d\right)\right|$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(b,c\right)\right|+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(b,d\right)\right|$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a,c\right)\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(a,d\right)\right|$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(b,c\right)\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f\left(b,d\right)\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(a,c\right)\right|+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(a,d\right)\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(b,c\right)\right|+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(b,d\right)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(a,c\right)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(a,d\right)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(b,c\right)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f\left(b,d\right)\right|\Bigg]$$

$$-\frac{1}{[3]_{p_1,q_1}[3]_{p_2,q_2}}$$

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$$\times\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|\right.$$

$$\left.+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|\right]$$

$$-\frac{[3]_{p_2,q_2}-[2]_{p_2,q_2}}{[3]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_2,q_2}}$$

$$\times\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|\right.$$

$$\left.+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|\right]$$

$$-\frac{[3]_{p_1,q_1}-[2]_{p_1,q_1}}{[2]_{p_1,q_1}[3]_{p_1,q_1}[3]_{p_2,q_2}}$$

$$\times\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|\right.$$

$$\left.+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|\right]$$

$$-\frac{\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}$$

$$\times\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|\right.$$

$$\left.\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|\right].$$

proof:

From the result of lemma 3.1 and Jensen-Mercer inequality, we have

$$\left|{}^{b,d}_{a,c}J_{p_1,p_2,q_1,q_2}f(t,s)\right|$$

$$\leq\int_0^1\int_0^1 ts\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|\right.$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|$$

$$-ts\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|-t(1-s)\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|$$

$$\left.-(1-t)s\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|-(1-t)(1-s)\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|\right]d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$+\int_0^1\int_0^1 ts\left[\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|\right.$$

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$$+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|$$

$$-(1-t)s\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|-(1-t)(1-s)\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|$$

$$-ts\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|-t(1-s)\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|\Bigg]d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$+\int_0^1\int_0^1 ts\Bigg[\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|$$

$$-t(1-s)\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|-ts\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|$$

$$-(1-t)(1-s)\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|-(1-t)s\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|\Bigg]d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$+\int_0^1\int_0^1 ts\Bigg[\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|$$

$$-(1-t)(1-s)\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|$$

$$-(1-t)s\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|$$

$$-t(1-s)\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|$$

$$-ts\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|\Bigg]d_{p_2,q_2}s\,d_{p_1,q_1}t$$

$$=\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\Bigg[\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|$$

$$+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|$$

$$+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|$$

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$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|\Bigg]$$

$$-\frac{1}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|\Bigg]$$

$$-\frac{[3]_{p_2,q_2}-[2]_{p_2,q_2}}{[3]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_2,q_2}}\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|$$

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$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|\Bigg]\Bigg]$$

$$-\frac{[3]_{p_1,q_1}-[2]_{p_1,q_1}}{[2]_{p_1,q_1}[3]_{p_1,q_1}[3]_{p_2,q_2}}\Bigg[\Bigg|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\Bigg|$$

$$+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|\Bigg]\Bigg]$$

$$-\frac{\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}$$

$$\times\Bigg[\Bigg|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\Bigg|+\Bigg|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\Bigg|$$

$$\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|\Bigg]\Bigg].$$

Corollary 3.4

If we set p_1, $p_2 = 1$, then the Theorem 3.2 reduces to the following inequality.

$$\left|{}_{a,c}^{b,d}J_{q_1,q_2}f(t,s)\right|$$

$$\leq\frac{1}{[2]_{q_1}[2]_{q_2}}\Bigg[\Bigg|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(a,c)\Bigg|+\Bigg|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(a,d)\Bigg|$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(b,c)\right|+\left|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(b,d)\right|$$

$$+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(a,c)\right|+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(b,c)\right|+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}_{c+d-l}\partial_{q_2}s}f(b,d)\right|$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}^{c+d-k}\partial_{q_2}s}f(a,c)\right|+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}^{c+d-k}\partial_{q_2}s}f(a,d)\right|$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}^{c+d-k}\partial_{q_2}s}f(b,c)\right|+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{q_1,q_2}}{{}_{a+b-n}\partial_{q_1}t\,{}^{c+d-k}\partial_{q_2}s}f(b,d)\right|$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2}}{{}^{a+b-m}\partial_{q_1}t\,{}^{c+d-k}\partial_{q_2}s}f(a,c)\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2}}{{}^{a+b-m}\partial_{q_1}t\,{}^{c+d-k}\partial_{q_2}s}f(a,d)\right|$$

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$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2} f(b,c)}{{}_{a+b-m}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2} f(b,d)}{{}_{a+b-m}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|\Bigg]\Bigg]$$

$$-\frac{1}{[3]_{q_1}[3]_{q_2}}\Bigg[\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2} f(m,k)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{q_1,q_2} f(n,k)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{q_1,q_2} f(m,l)}{{}_{a+b-n}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2} f(n,l)}{{}_{a+b-m}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|\Bigg]\Bigg]$$

$$-\frac{(q_2)^2}{[3]_{q_1}[2]_{q_2}[3]_{q_2}}\Bigg[\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2} f(m,l)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{q_1,q_2} f(n,l)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{q_1,q_2} f(m,k)}{{}_{a+b-n}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2} f(n,k)}{{}_{a+b-m}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|\Bigg]\Bigg]$$

$$-\frac{(q_1)^2}{[2]_{q_1}[3]_{q_1}[3]_{q_2}}\Bigg[\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2} f(n,k)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{q_1,q_2} f(m,k)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{q_1,q_2} f(n,l)}{{}_{a+b-n}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2} f(m,l)}{{}_{a+b-m}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|\Bigg]\Bigg]$$

$$-\frac{(q_1)^2(q_2)^2}{[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}}\Bigg[\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{q_1,q_2} f(n,l)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{q_1,q_2} f(m,l)}{{}_{a+b-n}\partial_{q_1} t\,{}_{c+d-l}\partial_{q_2} s}\right|$$

$$\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{q_1,q_2} f(n,k)}{{}_{a+b-n}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|+\left|\frac{{}^{a+b-m,c+d-k}\partial_{q_1,q_2} f(m,k)}{{}_{a+b-m}\partial_{q_1} t\,{}^{c+d-k}\partial_{q_2} s}\right|\Bigg]\Bigg].$$

Theorem 3.5

Suppose that tha assumptions of Lemma 3.1 are hold. If

$$\left|\frac{{}_{a,c}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1} x\,{}_{c}\partial_{p_2,q_2} y}\right|^{\beta},\left|\frac{{}^{b}_{c}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1} x\,{}_{c}\partial_{p_2,q_2} y}\right|^{\beta},\left|\frac{{}^{d}_{a}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1} x\,{}^{d}\partial_{p_2,q_2} y}\right|^{\beta},\left|\frac{{}^{b,d}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1} x\,{}^{d}\partial_{p_2,q_2} y}\right|^{\beta}$$

are coordinated convex function on $[a,b]\times[c,d]$, then we have the inequality

$$\left|{}^{b,d}_{a,c}J_{p_1,p_2,q_1,q_2} f(t,s)\right|$$

$$\leq\left(\frac{1}{[\alpha+1]_{p_1,q_1}[\alpha+1]_{p_2,q_2}}\right)^{\frac{1}{\alpha}}\Bigg\{\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,c)}{{}_{a+b-n}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,d)}{{}_{a+b-n}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,c)}{{}_{a+b-n}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,d)}{{}_{a+b-n}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,c)}{{}_{a+b-m}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,d)}{{}_{a+b-m}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,c)}{{}_{a+b-m}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,d)}{{}_{a+b-m}\partial_{p_1,q_1} t\,{}_{c+d-l}\partial_{p_2,q_2} s}\right|^{\beta}$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2} f(a,c)}{{}_{a+b-n}\partial_{p_1,q_1} t\,{}^{c+d-k}\partial_{p_2,q_2} s}\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2} f(a,d)}{{}_{a+b-n}\partial_{p_1,q_1} t\,{}^{c+d-k}\partial_{p_2,q_2} s}\right|^{\beta}$$

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$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$-\frac{1}{[\beta+1]_{p_1,q_1}[\beta+1]_{p_2,q_2}}\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}\right.$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}$$

$$\left.\left.+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}^{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\right]\right\}^{\frac{1}{\beta}}.$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1, \beta>1.$

Proof:

From the Lemma 3.1 and and Jensen-Mercer inequality by using the Hölder inequality and the convexity of $\left|\frac{{}_{a,c}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}_{a}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y}\right|^{\beta}, \left|\frac{{}^{b}_{c}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}^{b}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y}\right|^{\beta}, \left|\frac{{}^{d}_{a}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}_{a}\partial_{p_1,q_1}x^{d}\partial_{p_2,q_2}y}\right|^{\beta}, \left|\frac{{}^{b,d}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}^{b}\partial_{p_1,q_1}x^{d}\partial_{p_2,q_2}y}\right|^{\beta}$, we obtain

$$\left|{}^{b,d}_{a,c}J_{p_1,p_2,q_1,q_2}f(t,s)\right|$$

$$\leq\left(\int_0^1\int_0^1 t^{\alpha}s^{\alpha}d_{p_2,q_2}s\,d_{p_1,q_1}t\right)^{\frac{1}{\alpha}}\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}\right.$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}\left(\int_0^1\int_0^1 t^{\beta}s^{\beta}d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

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$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]$$

$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]$$

$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]^{\frac{1}{\beta}}$$

$$+\left(\int_0^1\int_0^1 t^{\alpha}s^{\alpha}d_{p_2,q_2}s d_{p_1,q_1}t\right)^{\frac{1}{\alpha}}\Bigg[\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]$$

$$-\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}\left(\int_0^1\int_0^1 t^{\beta}s^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\left(\int_0^1\int_0^1 t^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]^{\frac{1}{\beta}}$$

$$+\left(\int_0^1\int_0^1 t^{\alpha}s^{\alpha}d_{p_2,q_2}s d_{p_1,q_1}t\right)^{\frac{1}{\alpha}}\Bigg[\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$-\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}\left(\int_0^1\int_0^1 t^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}\left(\int_0^1\int_0^1 t^{\beta}s^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}(1-s)^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}s^{\beta}d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]^{\frac{1}{\beta}}$$

$$+\left(\int_0^1\int_0^1 t^{\alpha}s^{\alpha}d_{p_2,q_2}s d_{p_1,q_1}t\right)^{\frac{1}{\alpha}}\Bigg[\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

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$$-\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}(1-s)^{\beta}d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}\left(\int_0^1\int_0^1(1-t)^{\beta}s^{\beta}d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}\left(\int_0^1\int_0^1t^{\beta}(1-s)^{\beta}d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,=q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\left(\int_0^1\int_0^1t^{\beta}s^{\beta}d_{p_2,q_2}s\,d_{p_1,q_1}t\right)\Bigg]^{\frac{1}{\beta}}$$

$$=\left(\frac{1}{[\alpha+1]_{p_1,q_1}[\alpha+1]_{p_2,q_2}}\right)^{\frac{1}{\alpha}}\Bigg\{\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$+\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^{\beta}+\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^{\beta}$$

$$+\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{\beta}+\left|\frac{{}_{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\,{}^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{\beta}$$

$$-\frac{1}{[\beta+1]_{p_1,q_1}[\beta+1]_{p_2,q_2}}\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}\right.$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}$$

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$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^{\beta}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^{\beta}$$

$$\left.\left.+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^{\beta}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^{\beta}\right]\right\}^{\frac{1}{\beta}}.$$

Theorem 3.6

Suppose that tha assumptions of Lemma 3.1 are hold. If

$$\left|\frac{{}_{a,c}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}_{a}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y}\right|^{r},\left|\frac{{}_{c}^{b}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}^{b}\partial_{p_1,q_1}x\ {}_{c}\partial_{p_2,q_2}y}\right|^{r},\left|\frac{{}_{a}^{d}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}_{a}\partial_{p_1,q_1}x^{d}\partial_{p_2,q_2}y}\right|^{r},\left|\frac{{}^{b,d}\partial_{p_1,p_2,q_1,q_2}f(x,y)}{{}^{b}\partial_{p_1,q_1}x^{d}\partial_{p_2,q_2}y}\right|^{r}$$

are coordinated convex function on $[a,b]\times[c,d]$ for $r\geq 1$, then we have the inequality

$$\left|{}_{a,c}^{b,d}J_{p_1,p_2,q_1,q_2}f(t,s)\right|$$

$$\leq\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{r}}\left\{\left[\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^{r}+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^{r}\right.\right.$$

$$+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^{r}+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^{r}$$

$$+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^{r}+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^{r}$$

$$+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^{r}+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^{r}$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^{r}+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^{r}$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{r}+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{r}$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^{r}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^{r}$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^{r}+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^{r}$$

$$-\frac{1}{[r+1]_{p_1,q_1}[r+1]_{p_2,q_2}}\left[\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^{r}+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^{r}\right.$$

$$+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^{r}+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\ {}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^{r}$$

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$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(m,k)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(m,l)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(n,k)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(n,l)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2} f(m,k)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2} f(m,l)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r$$

$$+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2} f(n,k)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{c+d-k}_{a+b-n}\partial_{p_1,p_2,q_1,q_2} f(n,l)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2} f(m,k)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2} f(m,l)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r$$

$$\left.\left.+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2} f(n,k)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2} f(n,l)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}^{c+d-k}\partial_{p_2,q_2}s}\right|^r\right]\right]\right\}^{\frac{1}{r}}.$$

Proof:

From the Lemma 3.1 and and Jensen-Mercer inequality by using the power-mean inequality and the convexity of $\left|\frac{{}_{a,c}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_a\partial_{p_1,q_1}x\;{}_c\partial_{p_2,q_2}y}\right|^r,\left|\frac{{}^b_c\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^b\partial_{p_1,q_1}x\;{}_c\partial_{p_2,q_2}y}\right|^r,\left|\frac{{}^d_a\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}_a\partial_{p_1,q_1}x^d\partial_{p_2,q_2}y}\right|^r,\left|\frac{{}^{b,d}\partial_{p_1,p_2,q_1,q_2} f(x,y)}{{}^b\partial_{p_1,q_1}x^d\partial_{p_2,q_2}y}\right|^r$, we obtain

$$\left|{}^{b,d}_{a,c}J_{p_1,p_2,q_1,q_2} f(t,s)\right|$$

$$\le\left(\int_0^1\!\!\int_0^1 ts\,d_{p_2,q_2}s\,d_{p_1,q_1}t\right)^{1-\frac{1}{r}}\left[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,c)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,d)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r\right.$$

$$+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,c)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,d)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r$$

$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(m,k)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r\left(\int_0^1\!\!\int_0^1 t^r s^r d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(m,l)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r\left(\int_0^1\!\!\int_0^1 t^r(1-s)^r d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(n,k)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r\left(\int_0^1\!\!\int_0^1 (1-t)^r s^r d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

$$\left.-\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2} f(n,l)}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r\left(\int_0^1\!\!\int_0^1 (1-t)^r(1-s)^r d_{p_2,q_2}s\,d_{p_1,q_1}t\right)\right]^{\frac{1}{r}}$$

$$+\left(\int_0^1\!\!\int_0^1 ts\,d_{p_2,q_2}s\,d_{p_1,q_1}t\right)^{1-\frac{1}{r}}\left[\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,c)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(a,d)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r\right.$$

$$+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,c)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r+\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(b,d)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r$$

$$-\left|\frac{{}^{a+b-m}_{c+d-l}\partial_{p_1,p_2,q_1,q_2} f(m,k)}{{}^{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}\right|^r\left(\int_0^1\!\!\int_0^1 (1-t)^r s^r d_{p_2,q_2}s\,d_{p_1,q_1}t\right)$$

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$$-\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^r\left(\int_0^1\int_0^1(1-t)^r(1-s)^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^r\left(\int_0^1\int_0^1 t^r s^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^r\left(\int_0^1\int_0^1 t^r(1-s)^r d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]^{\frac{1}{r}}$$

$$+\left(\int_0^1\int_0^1 ts d_{p_2,q_2}s d_{p_1,q_1}t\right)^{1-\frac{1}{r}}\Bigg[\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^r+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^r$$

$$+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^r+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^r$$

$$-\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^r\left(\int_0^1\int_0^1 t^r(1-s)^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^r\left(\int_0^1\int_0^1 t^r s^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^r\left(\int_0^1\int_0^1(1-t)^r(1-s)^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^r\left(\int_0^1\int_0^1(1-t)^r s^r d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]^{\frac{1}{r}}$$

$$+\left(\int_0^1\int_0^1 ts d_{p_2,q_2}s d_{p_1,q_1}t\right)^{1-\frac{1}{r}}\Bigg[\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^r$$

$$+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^r$$

$$-\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^r\left(\int_0^1\int_0^1(1-t)^r(1-s)^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^r\left(\int_0^1\int_0^1(1-t)^r s^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^r\left(\int_0^1\int_0^1 t^r(1-s)^r d_{p_2,q_2}s d_{p_1,q_1}t\right)$$

$$-\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,=q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^r\left(\int_0^1\int_0^1 t^r s^r d_{p_2,q_2}s d_{p_1,q_1}t\right)\Bigg]^{\frac{1}{r}}$$

$$=\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{r}}\Bigg\{\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^r+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^r$$

$$+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^r+\left|\frac{{}^{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^r$$

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$$
\begin{aligned}
&+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,c)\right|^r+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(a,d)\right|^r\\
&+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,c)\right|^r+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(b,d)\right|^r\\
&+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^r+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^r\\
&+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^r+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^r\\
&+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,c)\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(a,d)\right|^r\\
&+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,c)\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(b,d)\right|^r\\
&-\frac{1}{[r+1]_{p_1,q_1}[r+1]_{p_2,q_2}}\Bigg[\Bigg[\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^r+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^r\\
&+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^r+\left|\frac{{}_{a+b-n,c+d-l}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^r\\
&+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,k)\right|^r+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(m,l)\right|^r\\
&+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,k)\right|^r+\left|\frac{{}_{c+d-l}^{a+b-m}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t\;{}_{c+d-l}\partial_{p_2,q_2}s}f(n,l)\right|^r\\
&+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^r+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^r\\
&+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^r+\left|\frac{{}_{a+b-n}^{c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-n}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^r\\
&+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,k)\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(m,l)\right|^r\\
&+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,k)\right|^r+\left|\frac{{}^{a+b-m,c+d-k}\partial_{p_1,p_2,q_1,q_2}}{{}_{a+b-m}\partial_{p_1,q_1}t^{c+d-k}\partial_{p_2,q_2}s}f(n,l)\right|^r\Bigg]\Bigg]\Bigg\}^{\frac{1}{r}}.
\end{aligned}
$$

Conclusion

In this paper, we have established several new Hermite-Hadamard-Mercer type inequalities on coordinates using post-quantum (p, q)-calculus. These results generalize existing inequalities and provide a unified framework encompassing both classical and quantum cases. Our work extends the literature by presenting double-integral identities under (p_1, p_2, q_1, q_2) - partial differentiability and integrability assumptions.Further research may explore: Higher-dimensional generalizations, Inequalities for other convexity types (e.g., s-convex, log-convex) and the applications in optimization or quantum information theory.

Data availability

All data required for this paper is included within this paper.

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References

1. Mercer AM. A variant of Jensen's inequality. J Inequal Pure Appl Math. 2003;4:1–6. Available from: https://eudml.org/doc/123826
2. Kian M, Moslehian MS. Refinements of the operator Jensen-Mercer inequality. Electron J Linear Algebra. 2013;26:742–753. Available from: https://journals.uwyo.edu/index.php/ela/article/view/1303/1303
3. Ali MA, Budak H, Abbas HM, Chu YM. Quantum Hermite-Hadamard-type inequalities for functions with convex absolute value of second derivatives. Adv Differ Equ. 2021;2021:1–12. Available from: https://advancesincontinuousanddiscretemodels.springeropen.com/articles/10.1186/s13662-020-03163-1
4. Ali MA, Chu YM, Budak H, Akkurt A, Yildirim H, Zahid MA. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. Adv Differ Equ. 2021;2021:25. Available from: https://doi.org/10.1186/s13662-020-03195-7
5. Aljinović AA, Kovačević D, Puljiz M, Keko AZ. On Ostrowski inequality for quantum calculus. Appl Math Comput. 2021;410:1264454. Available from: https://doi.org/10.1016/j.amc.2021.126454
6. Budak H, Ali MA, Tunç T. Quantum Ostrowski-type integral inequalities for functions of two variables. Math Methods Appl Sci. 2020;:5857–5872. Available from: https://doi.org/10.1002/mma.7153
7. Butt SI, Aftab MN, Nabwey HA, Etemad S. Some Hermite-Hadamard and midpoint type inequalities in symmetric quantum calculus. AIMS Math. 2024;9(3):5523–5549. Available from: https://www.aimspress.com/article/doi/10.3934/math.2024268
8. Butt SI, Umar M, Budak H. New study on the quantum midpoint-type inequalities for twice q-differentiable functions via the Jensen-Mercer inequality. Symmetry. 2023;15:1038. Available from: https://www.mdpi.com/2073-8994/15/5/1038
9. Ciurdariu L, Grecu E. Several quantum Hermite-Hadamard-type integral inequalities for convex functions. Fractal Fract. 2023;7:463. Available from: https://doi.org/10.3390/fractalfract7060463
10. Kalsoom H, Vivas-Cortez M. q1q2-Ostrowski-type integral inequalities involving property of generalized higher-order strongly n-polynomial preinvexity. Symmetry. 2022;14:717. Available from: https://doi.org/10.3390/sym14040717
11. Kalsoom H, Idrees M, Baleanu D, Chu YM. New estimates of q1q2-Ostrowski-type integral inequalities within a class of n-polynomial preinvexity of functions. J Funct Spaces. 2020;2020:Article ID 3720798. Available from: https://doi.org/10.1155/2020/3720798
12. Wannalookkhee F, Nonlaopon K, Ntouyas S, Sarikaya MZ, Budak H, Ali MA. Some new quantum Hermite-Hadamard inequalities for co-ordinated convex functions. Mathematics. 2022;10:1962. Available from: https://doi.org/10.3390/math10121962
13. Sarikaya MZ, Ertugral F. On the generalized Hermite-Hadamard inequalities. Ann Univ Craiova Math Comp Sci Ser. 2020;47(1):193–213. Available from: http://inf.ucv.ro/~ami/index.php/ami/article/viewFile/1139/724
14. Ali MAA, Budak H, Kalsoom H, Chu YM. Post-quantum Hermite-Hadamard inequalities involving newly defined (p,q)-integral. Authorea. 2020;1–19. Available from: https://doi.org/10.22541/au.160465507.75463188/v1
15. Ali MAA, Budak H, Myrtaza G, Chu YM. Post-quantum Hermite-Hadamard type inequalities for interval-valued convex functions. J Inequal Appl. 2021;2021:84. Available from: https://doi.org/10.1186/s13660-021-02619-6
16. Chu YM, Awan MU, Talib S, Noor MA, Noor KI. New post quantum analogues of Ostrowski-type inequalities using new definition/s of left-right (p,q)-derivatives and definite integrals. Adv Differ Equ. 2020;2020:634. Available from: https://doi.org/10.1186/s13662-020-03094-x
17. Gulshan G, Hussain R, Ali A. Post quantum-Hermite-Hadamard type inequalities for differentiable convex functions involving the notion of $(p,q)^b$ -integral. J Sci Arts. 2022;22(1):45–54. Available from: https://www.josa.ro/docs/josa_2022_1/a_04_Gulshan_45-54_10p.pdf
18. Kalsoom H, Amer M, Junjua MD, Hassain S, Shahzadi G. Some (p,q)-estimates of Hermite-Hadamard-type inequalities for coordinated convex and quasi convex functions. Mathematics. 2019;7:683. Available from: https://doi.org/10.3390/math7080683
19. Kunt M, Isan I, Alp N, Sarikaya MZ. (p,q)-Hermite-Hadamard inequalities and (p,q)-estimates for midpoint type inequalities via convex and quasi-convex functions. Rev Real Acad Cienc Exactas Fis Nat Ser A Mat. 2018;112:969–992. Available from: http://dx.doi.org/10.1007/s13398-017-0402-y
20. Luangboon W, Nonlaopon K, Tariboon J, Ntouyas SK, Budak H. Post-quantum Ostrowski-type integral inequalities for twice (p,q)-differentiable functions. J Math Inequal. 2022;16(3):1129–1144. Available from: https://jmi.ele-math.com/16-75/Post-quantum-Ostrowski-type-integral-inequalities-for-twice-(p,q)-differentiable-functions
21. Nasiruzzaman MD, Mukheimer A, Mursaleen M. Some Opial-type integral inequalities via (p,q)-calculus. J Inequal Appl. 2019;2019:1–11. Available from: https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-019-2247-8

22. Prabseang J, Nonlaopon K, Tariboon J. (p,q)-Hermite-Hadamard inequalities for double integral and (p,q)-differentiable convex function. Axioms. 2019;8:1–10. Available from: https://www.researchgate.net/publication/333438671_p_q-Hermite-Hadamard_Inequalities_for_Double_Integral_and_p_q-Differentiable_Convex_Functions

23. Sitthiwirattham T, Murtaza G, Ali MA, Ntouyas SK, Adeel M, Soontharanon J. On some new trapezoidal type inequalities for twice (p,q)-differentiable convex functions in post-quantum calculus. Symmetry. 2021;13:1605. Available from: https://doi.org/10.3390/sym13091605

24. Wannalookkhee F, Nonlaopon K, Tariboon J, Ntouyas SK. On Hermite-Hadamard type inequalities for coordinated convex functions via (p,q)-calculus. Mathematics. 2021;9:698. Available from: https://doi.org/10.3390/math9070698

25. Wannalookkhee F, Nonlaopon K, Ntouyas K, Sarikaya MZ, Budak H. Post quantum Ostrowski-type inequalities for coordinated convex functions. Math Methods Appl Sci. 2023;46:4159–4183. Available from: https://doi.org/10.1002/mma.8748

26. Sadjang PN. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. Results Math. 2016;73:1–21. Available from: https://www.researchgate.net/publication/256606098_On_the_Fundamental_Theorem_of_pq_-Calculus_and_Some_pq_-Taylor_Formulas

27. Tunc M, Göv E. (p,q)-integral inequalities. RGMIA Res Rep Coll. 2016;19:1–13. Available from: https://rgmia.org/papers/v19/v19a97.pdf

28. Kalsoom H, Rashid S, Idrees M, Safda F, Akram S, Baleanu D, et al. Post quantum inequalities of Hermite-Hadamard type associated with co-ordinated higher-order generalized strong pre-invex and quasi-pre-invex mapping. Symmetry. 2020;12:443. Available from: http://dx.doi.org/10.3390/sym12030443

29. Ali MM, Köbis E. Some new q-Hermite-Hadamard-Mercer inequalities and related estimates in quantum calculus. J Nonlinear Var Anal. 2023;7(1):49–66. Available from: https://jnva.biemdas.com/archives/1880

30. Araci S, Duran U, Acikgoz M, Srivastava HM. A certain (p,q)-derivative operator and associated divided differences. J Inequal Appl. 2016;2016:301. Available from: https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-016-1240-8

31. Awan MU, Talib S, Noor MA, Noor KI, Chu YM. On post quantum integral inequalities. J Math Inequal. 2021;15(2):629–654. Available from: https://jmi.ele-math.com/15-46/On-post-quantum-integral-inequalities

32. Awan MU, Talib S, Noor MA, Chu YM, Noor KI. On post quantum estimates of upper bounds involving twice (p,q)-differentiable preinvex function. J Inequal Appl. 2020;2020:229. Available from: https://doi.org/10.1186/s13660-020-02496-5

33. Bin-Mohsin B, Saba M, Javed MZ, Awan MU, Budak H, Nonlaopon K. A quantum calculus view of Hermite-Hadamard-Jensen-Mercer inequalities with applications. Symmetry. 2022;14:1246. Available from: https://doi.org/10.3390/sym14061246

34. Budak H, Pehlivan E, Kösem P. On new extensions of Hermite-Hadamard inequalities for generalized fractional integrals. Sahand Commun Math Anal. 2021;18(1):73–88. Available from: https://doi.org/10.22130/scma.2020.121963.759

35. Ciurdariu L, Grecu E. Post-quantum integral inequalities for three-times (p,q)-differentiable functions. Symmetry. 2023;15:246. Available from: https://doi.org/10.3390/sym15010246

36. Dragomir SS. On Hadamard inequality for convex functions on the coordinates in a rectangle from the plane. Taiwan J Math. 2001;4:775–788. Available from: https://www.jstor.org/stable/43834484

37. Duran U, Acikgoz M, Esi A, Araci S. A note on the (p,q)-Hermite-Hadamard polynomials. Appl Math Inf Sci. 2018;12:227–231. Available from: https://scispace.com/pdf/a-note-on-the-p-q-hermite-polynomials-2n7qwl6ufz.pdf

38. Ernst T. A comprehensive treatment of q-calculus. Basel: Springer; 2002. Available from: https://link.springer.com/book/10.1007/978-3-0348-0431-8

39. Kac V, Cheung P. Quantum calculus. New York: Springer; 2002. Available from: https://cdn.preterhuman.net/texts/math/quantum_calculus.pdf

40. Latif MA, Kunt M, Dragomir SS, Iscan I. Post-quantum trapezoid type inequalities. AIMS Math. 2020;5:4011–4026. Available from: https://doi.org/10.3934/math.2020258

41. Latif MA, Kunt M, Dragomir SS, Iscan I. (p,q)-trapezoid type inequalities. Rev R Acad Cienc Exactas Fís Nat Ser A Mat. 2018;112:969–992.

42. Li C, Yang D, Bai SC. Some Opial inequalities in (p,q)-calculus. AIMS Math. 2020;5:5893–5902. Available from: https://doi.org/10.3934/math.2020377

43. Liko R, Kashuri A. Some new refinement post quantum trapezium like inequalities pertaining twice (p,q)-differentiable convex functions. J Appl Math Comput Appl. 2021;2(1):58–85.

44. Luangboon W, Nonlaopon K, Tariboon J, Ntouyas SK, Budak H. Some (p,q)-integral inequalities of Hermite-Hadamard inequalities for (p,q)-differentiable convex functions. Mathematics. 2022;10:826. Available from: https://doi.org/10.3390/math10050826

45. Neang P, Nonlaopon K, Tariboon J, Ntouyas SK. Fractional (p,q)-calculus on finite intervals and some integral inequalities. Symmetry. 2021;13:504. Available from: https://doi.org/10.3390/sym13030504

46. Neang P, Nonlaopon K, Tariboon J, Ntouyas SK, Agarwal P. Some trapezoid and midpoint type inequalities via fractional (p,q)-calculus. Adv Differ Equ. 2021;2021:333. Available from: https://doi.org/10.1186/s13662-021-03487-6

47. Sadjang PN. On the (p,q)-gamma and (p,q)-beta functions. arXiv. 2015. Available from: https://arxiv.org/abs/1506.07394

48. Sarikaya MZ. On the Hermite–Hadamard type inequalities for co-ordinated convex function via fractional integrals. Integr Transf Spec Funct. 2014;25(2):134–147. Available from: http://dx.doi.org/10.1080/10652469.2013.824436

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49. Sontharanon J, Sitthiwirattham T. Fractional (p,q)-calculus. Adv Differ Equ. 2020;2020:1–18. Available from: https://advancesincontinuousanddiscretemodels.springeropen.com/articles/10.1186/s13662-020-2512-7

50. Thongjob S, Nonlaopon K, Ntouyas SK. Some (p,q)-Hardy type inequalities for (p,q)-integrable functions. AIMS Math. 2020;6:77–89. Available from: https://doi.org/10.3934/math.2021006

51. Tunc M, Göv E. (p,q)-integral inequalities via (p,q)-calculus on finite intervals. RGMIA Res Rep Coll. 2016;19:1–12. Available from: https://rgmia.org/papers/v19/v19a97.pdf

52. Vivas-Cortez M, Ali MA, Budak H, Kalsoom H, Agarwal P. Some new Hermite-Hadamard and related inequalities for convex functions via (p,q)-integral. Entropy. 2021;23(7):828. Available from: https://doi.org/10.3390/e23070828

53. Yu X, Ali MA, Kalsoom H, Soontharanon J, Sitthiwirattham T. On some new Hermite-Hadamard and Ostrowski type inequalities for s-convex functions in (p,q)-calculus with applications. De Gruyter Math. 2020;20:707–723. Available from: http://dx.doi.org/10.1515/math-2022-0037