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Short Communication

A Uniqueness Result for a Coupled System of Elliptic PDEs

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Abstract

Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary S,

$$Lu + av = 0, \quad -bu + Mv = 0 \text{ in } D,$$

$$u = v = 0 \text{ on } S.$$

Assume that L and M are positive elliptic Dirichlet operators of second order, $a > 0$ and $b > 0$ are constants. We prove that under these assumptions, the unique solution is $u = v = 0$ in D.

1. Introduction

This paper offers a partial solution to Open Problem 2 from [1] is given. In [2] and [3] the millennium problem related to the Navier-Stokes equations is solved.

Let D be a bounded domain in R^3 with smooth boundary S.

$$Lu + av = 0 \text{ in } D, \tag{1}$$

$$u = v = 0 \text{ on } S. \tag{2}$$

$$u = v = 0 \text{ on } S, \tag{3}$$

Assume that L and M are second order Dirichlet elliptic operators with real-valued coefficients, the bilinear form (Lu, u) is positive-definite for $u \neq o$.

$$(Lu, u) > 0 \text{ for } u \not\equiv 0, \tag{4}$$

$$(Mv, v) > 0 \text{ for } v \not\equiv 0, \tag{5}$$

a and b are constants,

$$a > 0, b > 0. \tag{6}$$

Theorem 1. *If assumptions (1)–(6) hold, then $u = v = 0$ in D.*

This problem concerns the uniqueness of solutions to coupled elliptic PDE systems with variable coefficients, a fundamental topic in mathematical analysis [1]. The full problem allows $a(x)$, $b(x)$ to vary spatially, increasing its generality and complexity. In the general formulation, the coupling coefficients $a(x)$ and $b(x)$ are positive continuous functions. $a = a(x) > 0$ and $b = b(x) > 0$.

2. Proofs

Lets define the inner product (u, v) over the domain D as $(u, v) := \int_D uv dx$. From our assumptions one derives, multiplying (1) by u and integrating by parts, the relation:

$$(Lu, u) + a(v, u) = 0, \tag{7}$$

and multiplying (2) by v and integrating by parts, the relation:

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$$-b(u,v)+(Mv,v)=0. \quad (8)$$

Combining the previous inequalities and assumptions yields from the assumptions (4)–(6) and from the inequalities (7) and (8) one gets, which can lead to the inequality:

$$(v,u)\le 0,\ (u,v)\ge 0. \quad (9)$$

As u and v are real-valued functions, it follows that, one has

$$(v,u)=(u,v). \quad (10)$$

Therefore

$$(v,u)=(u,v)=0 \quad (11)$$

Consequently,

$$(Lu,u)=(Mv,v)=0. \quad (12)$$

From (12) and our assumptions (4)–(5) it follows that $u = v = 0$ in D. Theorem 1 is proved. □

Remark. Note that in Theorem 1 there are no restrictions on the size of the constants a and b or $|a-b|$.

It is possible to show that the solution to problem (1)–(5) with $a = a(x)$ and $b = b(x)$, where $a(x)$ and $b(x)$ are continuous functions in the closure of D and

$$\sup_{x\in D}|a(x)-b(x)|<\nu, \quad (13)$$

where ν is a sufficiently small constant, is equal to zero in D.

Let us prove this conclusion. From (1)–(2) it follows that

$$(Lu,u)+\int_S(a(x)-b(x))u(x)v(x)dx+(Mv,v)=0. \quad (14)$$

From (4)–(5) it follows that

$$(Lu,u)>c_L\|u\|^2,\ (Mv,v)c_M\|v\|^2, \quad (15)$$

where $\|u\|^2=(u,u)$.

From our assumption (13) it follows that

$$|\int_S(a(x)-b(x))u(x)v(x)dx|\le\nu\|u\|\|v\|. \quad (16)$$

From (14)–(16) one derives:

$$0>c_L\|u\|^2+c_M\|v\|^2-\nu\|u\|\|v\|. \quad (17)$$

Choosing $\nu<c_L+c_M$, ensures that inequality (17) leads to $\|u\|=\|v\|=0$, so $u=v=0$ in D. □

Open problem: We pose the following open question: *Does the conclusion of Theorem 1 remain valid if the assumption that a,b are constants is replaced by the assumption that* $a=a(x)\ge 0$ *and* $b=b(x)\ge 0$ *are continuous functions?*

3. Conclusion

This result confirms that no non-trivial solution exists under constant coupling terms, contributing to PDE theory. Representing a coupled elliptic system with Dirichlet boundary conditions (1)–(3). The basic assumptions are: $a > 0$ and $b > 0$ are constants, L and M are positive elliptic operators such that $(Lu, u) = 0$ implies $u = 0$, and $(Mv, v) = 0$ implies $v = 0$.

This result contributes to the broader understanding of stability and uniqueness in PDE modeling across physics and engineering.

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