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## Short Communication

# A Uniqueness Result for a Coupled System of Elliptic PDEs

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## Abstract

Let  $D \subset \mathbb{R}^3$  be a bounded domain with a smooth boundary  $S$ ,

$$Lu + av = 0, \quad -bu + Mv = 0 \text{ in } D,$$

$$u = v = 0 \text{ on } S.$$

Assume that  $L$  and  $M$  are positive elliptic Dirichlet operators of second order,  $a > 0$  and  $b > 0$  are constants. We prove that under these assumptions, the unique solution is  $u = v = 0$  in  $D$ .

## 1. Introduction

This paper offers a partial solution to Open Problem 2 from [1] is given. In [2] and [3] the millennium problem related to the Navier-Stokes equations is solved.

Let  $D$  be a bounded domain in  $\mathbb{R}^3$  with smooth boundary  $S$ .

$$Lu + av = 0 \text{ in } D, \quad (1)$$

$$u = v = 0 \text{ on } S. \quad (2)$$

$$u = v = 0 \text{ on } S, \quad (3)$$

Assume that  $L$  and  $M$  are second order Dirichlet elliptic operators with real-valued coefficients, the bilinear form  $(Lu, u)$  is positive-definite for  $u \neq 0$ .

$$(Lu, u) > 0 \text{ for } u \neq 0, \quad (4)$$

$$(Mv, v) > 0 \text{ for } v \neq 0, \quad (5)$$

$a$  and  $b$  are constants,

$$a > 0, b > 0. \quad (6)$$

**Theorem 1.** If assumptions (1)–(6) hold, then  $u = v = 0$  in  $D$ .

This problem concerns the uniqueness of solutions to coupled elliptic PDE systems with variable coefficients, a fundamental topic in mathematical analysis [1]. The full problem allows  $a(x)$ ,  $b(x)$  to vary spatially, increasing its generality and complexity. In the general formulation, the coupling coefficients  $a(x)$  and  $b(x)$  are positive continuous functions.  $a = a(x) > 0$  and  $b = b(x) > 0$ .

## 2. Proofs

Lets define the inner product  $(u, v)$  over the domain  $D$  as  $(u, v) := \int_D uv dx$ . From our assumptions one derives, multiplying (1) by  $u$  and integrating by parts, the relation:

$$(Lu, u) + a(v, u) = 0, \quad (7)$$

and multiplying (2) by  $v$  and integrating by parts, the relation:

$$-b(u, v) + (Mv, v) = 0. \quad (8)$$

Combining the previous inequalities and assumptions yields from the assumptions (4)–(6) and from the inequalities (7) and (8) one gets, which can lead to the inequality:

$$(v, u) \leq 0, (u, v) \geq 0. \quad (9)$$

As  $u$  and  $v$  are real-valued functions, it follows that, one has

$$(v, u) = (u, v). \quad (10)$$

Therefore

$$(v, u) = (u, v) = 0 \quad (11)$$

Consequently,

$$(Lu, u) = (Mv, v) = 0. \quad (12)$$

From (12) and our assumptions (4)–(5) it follows that  $u = v = 0$  in  $D$ . Theorem 1 is proved.  $\square$

**Remark.** Note that in Theorem 1 there are no restrictions on the size of the constants  $a$  and  $b$  or  $|a - b|$ .

It is possible to show that the solution to problem (1)–(5) with  $a = a(x)$  and  $b = b(x)$ , where  $a(x)$  and  $b(x)$  are continuous functions in the closure of  $D$  and

$$\sup_{x \in D} |a(x) - b(x)| < \nu, \quad (13)$$

where  $\nu$  is a sufficiently small constant, is equal to zero in  $D$ .

Let us prove this conclusion. From (1)–(2) it follows that

$$(Lu, u) + \int_S (a(x) - b(x))u(x)v(x)dx + (Mv, v) = 0. \quad (14)$$

From (4)–(5) it follows that

$$(Lu, u) > c_L \|u\|^2, (Mv, v) < c_M \|v\|^2, \quad (15)$$

where  $\|u\|^2 = (u, u)$ .

From our assumption (13) it follows that

$$|\int_S (a(x) - b(x))u(x)v(x)dx| \leq \nu \|u\| \|v\|. \quad (16)$$

From (14)–(16) one derives:

$$0 > c_L \|u\|^2 + c_M \|v\|^2 - \nu \|u\| \|v\|. \quad (17)$$

Choosing  $\nu < c_L + c_M$ , ensures that inequality (17) leads to  $\|u\| = \|v\| = 0$ , so  $u = v = 0$  in  $D$ .  $\square$

**Open problem:** We pose the following open question: Does the conclusion of Theorem 1 remain valid if the assumption that  $a, b$  are constants is replaced by the assumption that  $a = a(x) \geq 0$  and  $b = b(x) \geq 0$  are continuous functions?

### 3. Conclusion

This result confirms that no non-trivial solution exists under constant coupling terms, contributing to PDE theory. Representing a coupled elliptic system with Dirichlet boundary conditions (1)–(3). The basic assumptions are:  $a > 0$  and  $b > 0$  are constants,  $L$  and  $M$  are positive elliptic operators such that  $(Lu, u) = 0$  implies  $u = 0$ , and  $(Mv, v) = 0$  implies  $v = 0$ .

This result contributes to the broader understanding of stability and uniqueness in PDE modeling across physics and engineering.

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