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Research Article

Common Fixed Point Theorems via Measure of Noncompactness

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Abstract

In this paper, by applying the measure of noncompactness a common fixed point for the maps T and S is obtained, where T and S are self-maps continuous, commuting continuously on a closed convex subset C of a Banach space E and also S is a linear map. Then as an application, the existence of a solution of an integral equation is shown.

1. Introduction

The compactness plays an essential role in the Schauder's fixed point theorem and however, there are some important problems where the operators are not compact. G. Darbo in 1955 [1], extended the Schauder theorem to noncompact operators . The main aim of their study is to define a new class of operators that map any bounded set to a compact set. The first measure of noncompactness was defined and studied by Kuratowski [2] in 1930.

Suppose (X, d) be a metric space the Kuratowski measure of noncompactness of a subset

$A \subset X$ defined as

$$\mu(A) = inf\left\{\delta > 0; A = \cup_{i=1}^{n} A_i \, for \, some \, A_i \, with \, diam\left(A_i\right) \le \delta \, for \, 1 \le i \le n < \infty\right\} \quad (1)$$

where $diam\ (A)$ denotes the diameter of a set $A \subset X$ namely

$$diam(A) = \sup\{d(x,y); x, y \in A\}.$$

Also, in recent years measures of noncompactness have been used to define new geometrical properties of Banach spaces which are interesting for fixed point theory [3]. In this paper first, some essential concepts and results concerning the measure of noncompactness are called [4-7]. In the second section, a common fixed point for the maps T and S where T and S are self-map continuous, commuting continuous on a closed convex subset C of a Banach space E and also S is a linear map is shown. In the third section, we apply our result to obtain a coupled fixed point [8-11] . Finally by applying our results a solution of an integral equation is obtained [12-15].

Now, we recall some basic facts concerning measures of noncompactness. Suppose R denotes the set of real numbers and put $R_+ = [0, \infty)$ and let $(E, \|.\|)$ be a Banach space. The symbol $\overline{X}, Conv\,X$ will denote the closure and closed convex hull of a subset X of E , respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets.

We begin by recalling some needed definitions and results.

Definition 1.1 A mapping $\mu:\mathfrak{M}_E \to R_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1. The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subseteq \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
3. $\mu(\bar{X}) = \mu(X)$.
4. $\mu(Conv\,X) = \mu(X)$.
5. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\,\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
6. If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$, and if $\lim_{n\to\infty}\mu(X_n) = 0$, then

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \ne 0$$

Theorem 1.1 (Schauder [9]) Let C be a closed and convex subset of a Banach space E. Then every compact and continuous map $F: C \to C$ has at least one fixed point.

In 1955, G. Darbo [1] used the measure of noncompactness to generalize Schauder's theorem to a wide class of operators, called k-set contractive operators, which satisfy the following condition

$$\mu(T(A)) \le k\mu(A)$$

for some $k \in [0,1)$. In 1967 Sadovskii generalized Darbo's theorem to set-condensing operators [16,17].

Definition 1.2 Let E_1 and E_2 be two Banach spaces and μ_1 and μ_2 be arbitrary measures of noncompactness on E_1 and E_2 respectively [5] . An operator T from E_1 to E_2 is called a (μ_1, μ_2) condensing operator if it is continuous and for every bounded noncompact set $\Omega \subset E_1$ the following inequality holds

$$\mu_2(T(\Omega)) < \mu_1(\Omega).$$

The following lemmas and theorems from [16-18] are necessary for the main results.

Theorem 1.2 (Darbo's fixed point theorem) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T:\Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0,1)$ with the property [18].

$$\mu(TX) \le k\mu(X);$$

For any nonempty subset X of Ω Then T has a fixed point in the set Ω.

Lemma 1.3 For every nondecreasing and upper semicontinuous function $\varphi: R_+ \to R_+$ The following two conditions are equivalent:

i. $\lim_{n\to\infty}\varphi^n(t) = 0$ for any $t > 0$.

ii. $\varphi(t) < t$ for any $t > 0$.

The following theorem is an extension of Darbo's fixed point theorem.

Theorem 1.4 [7] Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T: C \to C$ be a continuous operator satisfying

$$\mu(T(X)) \le \varphi(\mu(X)) \tag{2}$$

for any subset X of C, where μ is an arbitrary measure of noncompactness and $\varphi: R_+ \to R_+$ is a nondecreasing and upper semicontinuous function such that $\varphi(t) < t$ for all $t > 0$. Then T has at least one fixed point.

2. Common fixed point

Theorem 2.1 Let C be a nonempty, bounded, closed, and convex subset of a Banach space E.

and let $T, s: C \to C$ be continuous operators and S be a linear operator such that $S(T(X)) \subseteq T(X)$ and also

$$\mu(T(X)) \le \varphi(\max\{\mu(X), \mu(S(X))\}),$$

for each $X \subseteq C$, where μ is an arbitrary measure of noncompactness and $\varphi: R_+ \to R_+$ is a nondecreasing function such that $\varphi(t) < t$ for each $t > 0$ and $\varphi(0) = 0$. Then T, S have a common fixed point in C.

Proof. Set

$$C_0 = C$$

And

$$C_1 = Conv\,TC_0$$

in general, set

$$C_n = Conv\,TC_{n-1}$$

For n = 1,2, ...

Then we have

$$C_n \subset C_{n-1}\ and\ S(C_n) \subset C_n\ (*)$$

for ever n = 1,2,3, ...

Indeed it is clear that $C_1 \subset C_0$ and $S(C_1) \subset Conv(ST(C_0)) \subset Conv(T(C_0)) = C_1$.

So (∗) holds for n = 1.

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Assuming now that (*) is true for $n \geq 1$.

Then

$$C_{n+1} = Conv(T(C_n)) \subset Conv(T(T_{n-1})) = C_n$$

And

$$S(C_{n+1}) = S(Conv(T(C_n))) \subset Conv(S(T(C_n))) \subset ConvT(C_n) = C_{n+1}$$

We obtain

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$$

Now if there exists an integer $N \geq 0$ such that $\mu(C_N) = 0$, then C_N is relatively compact and since $TC_N \subseteq Conv TC_{N+1} \subseteq C_N$, thus Schauder's fixed point theorem implies that T has a fixed point. So we assume that $\mu(C_n) \geq 0$ for $n \geq 0$. By assumptions we have

$$\begin{aligned}\mu(C_{n+1}) &= \mu(Conv TC_n) \\ &= \mu(TC_n) \\ &\leq \varphi(\max\{\mu(TC_n), \mu(STC_n)\}) \\ &\leq \varphi(\mu(TC_n)) \\ &\leq \mu(TC_n) \\ &\leq \mu(C_n)\end{aligned}$$

which implies that $\mu(C_n)$ is a positive decreasing sequence of real numbers thus, there is an $r \geq 0$ so that $\mu(C_n) \to r$ as $n \to \infty$. We show that $r = 0$. Suppose, in the contrary, that $r \geq 0$. Then we have

$$\begin{aligned}\mu(C_{n+1}) &= \mu(Conv TC_n) \\ &= \mu(TC_n) \\ &\leq \varphi(\mu(TC_n)) \\ &\leq \varphi(\mu(C_n)) \\ &= \varphi(\mu(Conv TC_{n-1})) \\ &\leq \varphi(\mu(TC_{n-1})) \\ &\leq \varphi^2(\mu(C_{n-1})) \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \varphi^n(\mu(C_0)).\end{aligned}$$

By Lemma 1.3 and assumption with choose $\mu(C_0) = t$, we have

$$r = \lim_{n\to\infty} \mu(C_{n+1}) \leq \lim_{n\to\infty} \varphi^n(\mu(C_0)) = \lim_{n\to\infty} \varphi^n(t) = 0$$

for any $t > 0$.

So $r = 0$ and hence $\mu(C_n) \to 0$ as $n \to \infty$. Since $C_{n+1} \subseteq C_n$ and $TC_n \subseteq C_n$ for all $n \geq 1$, then from (6), $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is a nonempty convex closed set, and $C_\infty \subset C$. Moreover, the set C_∞ is invariant under the operator T and belongs to ker μ. Thus, applying Schauder's fixed point theorem, T has a fixed point. Now, suppose that $F_T = \{x \in C : Tx = x\}$. The set F_T is closed by the continuity of T, by the assumption we have $SF_T \subset F_T$ then Sx is a fixed point of T for any $x \in F_T$ and

$$\begin{aligned}\mu(F_T) &= \mu(TF_T) \leq \varphi(\max\{\mu(F_T), \mu(SF_T)\}) \\ &= \varphi(\mu(F_T)) \\ &< \mu(F_T)\end{aligned}$$

then $\mu(F_T) = 0$ and have F_T is compact.

Then by Schauder's fixed point theorem, we deduce that S has a fixed point and set $F(S) = \{x \in C, Sx = x\}$ is closed by the continuity of S. Also, since $SF_T \subset F_T$ by Schauder's fixed point theorem, we have Tx is a fixed point of S for each $x \in F_S$. Since $F_T \bigcap F_S \subseteq F_T \subset C$ is a compact subset, then $T, s : F_T \bigcap F_S \to F_T \bigcap F_S$ are continuous self maps, now by Schauder's fixed point theorem we have a common fixed point in C.

Corollary 2.2 Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T, S : C \to C$ be continuous operators and S be a linear operator such that

$$S(T(X)) \subseteq T(X)$$

and

$$\mu(TX) \leq k \max\{\mu(X), \mu(SX)\},$$

for each $X \subseteq C$, where μ is an arbitrary measure of noncompactness and $k \in [0,1)$. Then T, S have a common fixed point in C.

Proof. Let $\varphi(t) = kt$ in the Theorem 2.1 .

Corollary 2.3 Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T, S : C \to C$ be continuous operators and S be a linear and condensing operator such that.

$$S(T(X)) \subseteq T(X)$$

and

$$\mu(TX) \leq \varphi(\mu(X)),$$

For each $X \subseteq C$, where μ is an arbitrary measure of noncompactness and $\varphi R_+ \to R_+$ is a nondecreasing function such that $\varphi(t) < t$ for each $t \geq 0$ and $\varphi(0) = 0$. Then T, S have a common fixed point in C.

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Proof. The result is followed by Definition 1.2 and Theorem 2.1.

Corollary 2.4 Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T,S:C\to C$ be continuous operators and S be a linear operator such that T and S be two commuting map and

$$\mu(T(X))\le\varphi(max\{\mu(X),\mu(S(X))\}),$$

For each $X\subseteq C$, where μ is an arbitrary measure of noncompactness and $\varphi:R_+\to R_+$ is a nondecreasing function such that $\varphi(t)<t$ for each $t\ge 0$ and $\varphi(0)=0$. Then T, S have a common fixed point in C [19-23].

Proof. The proof is similar to the proof of Theorem 2.1.

Definition2.1 Let X be a Banach space. An operator (not necessarily linear) $F:X\to X$ is compact if the closure of $F(Y)$ is compact whenever $Y\subset X$ is bounded.

Corollary 2.5 Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $F:C\to E$ be a linear and continuous operator such that T and S be two commuting map and

$$\|Fx-Fy\|\le\varphi(\|x-y\|), \tag{3}$$

where $\varphi:R_+\to R_+$ is a nondecreasing function such that $\varphi(t)<t$ for each $t\ge 0$ and $\varphi(0)=0$. Assume that $G:C\to E$ is a compact, continuous operator. Define $T(x):=F(x)+G(x)$ and assume that $T(x)\in C$ for all $x\in C$. Then T, S have a common fixed point in C.

Proof. Let $\mu:\mathfrak{M}_E\to R_+$ be the Kuratowski measure of noncompactness defined by (1). Moreover, assume that X is a nonempty subset of C. As φ is non-decreasing, from (3), we have

$$\|Fx-Fy\|\le sup_{x,y\in X}\varphi(\|x-y\|)\le\varphi\left(sup_{x,y\in X}\|x-y\|\right),$$

so

$$diam(F(X))\le\varphi(diam(X)). \tag{4}$$

By the definition of Kuratowski measure of noncompactness, for every $\delta>0$, there exist $A_1,\ldots,A_n$ such that $X\subseteq\bigcup_{i=1}^n A_i$ and $diam(A_i)<\mu(X)+\delta$. As $F(X)\subseteq\bigcup_{i=1}^n F(A_i)$ and by assumption, φ is a non-decreasing function, from (4) we have

$$\mu(F(X))\le diam(F(A_i))\le(\varphi(diam(A_i))\le\varphi(\mu(X)+\delta)$$

and

$$\mu(F(X))\le\varphi(\mu(X)). \tag{5}$$

On the other hand, as G is compact, from (5) we obtain

$$\mu(T(X))=\mu((F+G)(X))\le\mu(F(X)+G(X))\le\mu(F(X))+\mu(G(X))\le\varphi(\mu(X)).$$

Now, by Theorem 1.4, T has a fixed point in C, Now, suppose that $F_T=\{x\in C:Tx=x\}$ is closed by the continuity of T.

On the other hand, since S commuting with T, we see that Sx is a fixed point of T for any $x\in F_T$.

Thus $SF_T\subset F_T$ and since

$$\mu(F_T)=\mu(TF_T)\le\varphi(\max\{\mu(F_T),\mu(SF_T)\})$$
$$=\varphi(\mu(F_T))$$
$$<\mu(F_T)$$

then $\mu(F_T)=0$ and have F_T is compact.

Then by Schauder's fixed point theorem, we deduce that S has a fixed point and set $F_S=\{x\in C,Sx=x\}$ is closed by the continuity of S. Also, since S commutes with T, we have Tx is a fixed point of S for each $x\in F_S$, therefore Fs is invariant by T or $T(F_S)\subset F_S$. Since Fs is convex closed and bounded and for any $D\subset F_S$ we have

$$\mu(T(D))\le\varphi(max\{\mu(D),\mu(S(D))\})$$
$$\le\varphi(\mu(D)).$$

Then by Corollary 2.4, T and S have a common fixed point in D.

Corollary 2.6 Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $S,G:C\to C$ be continuous operators and S be a linear operator and G be a compact operator, define $T(x):=S(x)+G(x)$ and assume that $T(x)\in C$ for all $x\in C$, such that T and S be two commuting map. Then T, S have a common fixed point in C [2,24-27].

Proof. Since G is a compact operator, we have $\mu(G(C))=0$ and so $\mu(T(C))=\mu(S(C))$ so T, S have a common fixed point in C.

Example 2.1 ([7]) Let *C[a,b]* denote the Banach space consisting of all real-valued functions, defined and continuous on *[a,b]*. The space *C[a,b]* is furnished with the standard norm

$$\|x\|=max\{|x(t)|:t\in[a,b]\}$$

for every $x\in C[a,b]$.

A measure of noncompactness can be defined as follows. To this end let us fix a nonempty bounded subset X of $C[a,b]$. For $x\in X$ and $\varepsilon>0$ let us denote by $\omega(x,\varepsilon)$ the modulus of continuity of the function x on the interval $[a,b]$, i.e

$$\omega(x,\varepsilon)=sup\{|x(t)-x(s)|:t,s\in[a,b],|t-s|<\varepsilon\}$$

$$\omega(X,\varepsilon)=sup\{\omega(x,\varepsilon):x\in X\}$$

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$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon)$$

and

$$X(t) = \{x(t) : x \in X\}.$$

It is easy to prove that ω_o is a measure of noncompactness and

$$\mu(X) = \frac{1}{2}\omega_0(X).$$

In the following, we will show some examples of the results.

Example2.2 Let $C = [0,1]$ a nonempty, bounded, closed, and convex subset of a Banach space $R, T, S : C \to C$ be continuous operators and S be a linear operator where

$$S(TX) \subseteq T(X)$$

and also

$$\mu(T(X)) \le \varphi\left(max\{\mu(X), \mu(S(X))\}\right),$$

For each $X \subseteq C$ is holed.

Then consider $Sx = \frac{x}{2}$, $Tx = \frac{x}{x+3}$ and $\varphi : R_+ \to R_+$ by $\varphi(t) = \frac{t}{2}$ is a nondecreasing function such that $\varphi(t) < t$ for each $t \ge 0$ and $\varphi(0) = 0$.

Let μ be the same measure of noncompactness in Example 2.1. Then by Theorem 2.1, *T*, *S* have a common fixed point $x = 0$ in *C*.

Example 2.3 Let *R* be a Banach space and we define:

$$T, S : C \to C$$

$$T(x) = \begin{cases} 0 & x \le 0 \\ x - \frac{x^2}{2} & 0 < x \le 1 \\ \frac{1}{2} & x > 1 \end{cases}$$

$$S(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$$

And

$$\varphi : R_+ \to R_+$$

by

$$\varphi(t) = \begin{cases} t - \frac{t^2}{2} & 0 < t \le 1 \\ \frac{t}{2} & t > 1 \end{cases}$$

is a non-decreasing function such that $\varphi(t) < t$ for each $t \ge 0$ and $\varphi(0) = 0$. Clearly *T*, *S* are commuting maps and with corresponding Corollary 2.4, for any subset $D \subset R$, obviously we have

$$\mu(TD) \le \varphi\left(max\{\mu(D), \mu(S(D))\}\right).$$

Then *T*, *S* have a common fixed point *x* = 0.

3. Common coupled fixed point

Definition 3.1 [8] An element $(X, Y) \in X \times X$ is called a coupled fixed point of the operator $F : X \times X \to X$ if $F(x, y) = x$ and $F(x, y) = y$.

Definition 3.2 The operators $T, S : C \times C \to C$ is called commuting operator if

$$T(S(x, y), S(y, x)) = S(T(x, y), T(y, x))$$

for all $x, y \in C$.

Theorem 3.1 [7] Suppose $\mu_1, \mu_2, \ldots, \mu_n$ be measures of noncompactness on, Banach spaces $E_1, E_2, \ldots, E_n$ respectively. Moreover assume that the function $F : R_+^n \to R_+$ is convex and $F(x_1, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$. Then

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness on $E_1 \times E_2 \times \ldots \times E_n$ where X_i denotes the natural projections of *X* into E_i for $i = 1, 2, \ldots, n$.

Remark 3.1 [4] Let μ be a measure of noncompactness on a Banach space *E* considering

$$F_1(x, y) = max\{x, y\} \text{ and } F_2(x, y) = x + y$$

for $(x, y) \in R_+^2$ then conditions of Theorem 3.1 are satisfied. Therefore,

$$\tilde{\mu}_1(X) := max\{\mu(X_1), \mu(X_2)\}$$

And

$$\tilde{\mu}_2(X) := \mu(\mu_1) + \mu(\mu_2)$$

Define measures of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural

projections of *X* into *E*.

Theorem 3.2 Let *C* be a nonempty, bounded, closed, and convex subset of a Banach space *E*

and let $T, S : C \times C \to C$ be continuous operators and *S* be a linear operator such that

$$S(T(X) \times T(Y)) \subseteq T(X) \times T(Y)$$

and

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$$\mu(T(X\times Y))\le\varphi(max\{\mu(X),\mu(Y),\mu(S(X\times Y))\}) \quad (6)$$

for each $X,Y\subseteq C$, where μ is an arbitrary measure of noncompactness and $\varphi:R_+\to R_+$ is a non-decreasing function such that $\varphi(t)<t$ for each $t\ge 0$ and $\varphi(0)=0$. Then T, S have a common coupled fixed point in C.

Proof. First note that, Remark 3.1 implies that

$$\tilde{\mu}(X)=\mu(X_1)+\mu(X_2)$$

is a measure of noncompactness in the space $E\times E$ where $X_i, i=1,2$ denote the natural projection of X. Now consider the map. $\tilde{G}:\Omega\times\Omega\to\Omega\times\Omega$ defined by the formula

$$\tilde{G}(x,y)=(G(x,y),G(y,x))$$

Which is continuous on $\Omega\times\Omega$. We claim that $\tilde{G}$ satisfies all the conditions of Theorem 2.1. To prove this, let $X\subset\Omega\times\Omega$ be a nonempty subset. Then, by 2º and (6) we have

$$\tilde{\mu}(\tilde{G}(X))\le\tilde{\mu}(G(X_1\times X_2)\times G(X_2\times X_1))$$
$$=\mu(G(X_1\times X_2))+\mu(G(X_2\times X_1))$$
$$\le\varphi(max\{\mu(X_1),\mu(X_2),\mu(S(X_1\times X_2))\})$$
$$+\varphi(max\{\mu(X_1),\mu(X_2),\mu(S(X_2\times X_1))\})$$
$$=2\varphi(max\{\mu(X_1),\mu(X_2),\mu(S(X_1\times X_2))\})$$

Then

$$\frac{1}{2}\tilde{\mu}(\tilde{G}(X))\le\varphi(max\{\mu(X_1),\mu(X_2),\mu(S(X_1\times X_2))\})$$
$$\le\varphi(max\{\mu(X_1)+\mu(X_2),\mu(S(X_1\times X_2))\})$$
$$\le\varphi(max\{2(\mu(X_1)+\mu(X_2)),\mu(S(X_1\times X_2))\})$$
$$=\varphi(max\{2\tilde{\mu}(X),\mu(S(X_1\times X_2))\})$$

and taking $\tilde{\mu}'=\frac{1}{2}\tilde{\mu}$, $\tilde{\mu}'(S(X))=\mu(S(X_1\times X_2))$ we get

$$\tilde{\mu}'(G(X))\le\varphi(max\{\tilde{\mu}'(X),\tilde{\mu}'(S(X))\})$$

Since, $\tilde{\mu}'$ is also a measure of noncompactness, therefore, all the conditions of Theorem 2.1 are satisfied and G has a coupled fixed point.

4. Application

Let $L^1(R_+)$ be the space of Lebesgue integrable functions on the measurable subset R_+ of R with the standard norm

$$\|x\|=\int_0^\infty|x(t)|dt.$$

Now, we define a measure of noncompactness in the space. $L^1(R_+)$.

For $\varepsilon>0$, let X be a nonempty, bounded, compact, and measurable subset of $L^1(R_+)$, set

$$C(X)=\lim_{\varepsilon\to 0}\sup_{x\in X}\sup\left\{\int_D|x(t)|\,dt: D\subset R_+; meas(D)\le\varepsilon\right\};$$

Where $means(D)$ denotes the Lebesgue measure of the subset D and

$$d(x)=\lim_{T\to\infty}sup\left\{\int_T^\infty|x(t)|dt: x\in X\right\}.$$

Then we define

$$\mu(X)=C(X)+d(X),$$

Where u is a measure of noncompactness in $L^1(R_+)$.

Our purpose is the study of the equation below:

$$x(t)=(1-\lambda)\int_0^\infty k(t-s)x(s)ds+\lambda f(t,\int_0^\infty k(t-s)x(\varphi(s))ds);t\ge 0,\lambda\in(0,1) \quad (7)$$

under the following hypotheses.

i. $f:R_+\times R\to R$ and there is a constant $0<b<1$ such that

$$f(t,x)=bx+\exp(-t).$$

ii. The function $k:R\to R_+$ belongs to the space $L^1(R_+)$, defined by $k(t)=\exp(-t)$ for $t\in[0,1]$ and $k(t)=0$ for $t<0$ and $t>1$.

Therefore, we can see that for any $A>0$ and for all $t_1,t_2\in R_+$ the following condition is satisfied:

$$t_1<t_2\Rightarrow\int_0^A k(t_2-s)ds\le\int_0^A k(t_1-s)ds.$$

iii. $\varphi:R_+\to R_+$ is a non-decreasing function such that $\varphi(t)<t$ for each $t>0$ and $\varphi(0)=0$.

iv. The linear continuous operator K is defined by

$$(Kx)(t)=\int_0^\infty k(t-s)x(s)ds$$

Maps Q_r into Q_r. (Let E be an arbitrary Banach space with norm $\|\cdot\|$ and the zero element 0 and B_r be a closed ball in E centered at 0 and of radius r, and also suppose Q_r be the subset of B_r consisting of all functions that are a.e. positive and nonincreasing on R_+, which is a compact, bounded, closed, and convex subset of $L^1(R_+)$.

Then we can prove the following result.

Theorem 4.1 Let the assumptions i), ii), iii), and iv) be satisfied. Then the equation (7) has at least one solution $x\in L^1(R_+)$ such that

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$$x(t)=\int_0^{\infty}k(t-s)x(s)ds.$$

Proof.

Step 1: We consider the following operators

$$(Hx)(t)=(1-\lambda)\int_0^{\infty}k(t-s)x(s)ds+\lambda f(t,\int_0^{\infty}k(t-s)x(\varphi(s))ds),$$

$$(Kx)(t)=\int_0^{\infty}k(t-s)x(s)ds$$

and

$$(Fx)(t)=f(t,x(t)).$$

Thus the equation (7) becomes

$$x=Hx=(1-\lambda)Kx+\lambda FKx(\varphi).$$

Next, we consider

$$Gx=\frac{Hx-(1-\lambda)Kx}{\lambda}=FKx(\varphi),$$

Step 2: For any $x\in L^1(R_+)$ we have

$$\|Gx\|=\|FKx(\varphi)\|$$

$$=\int_0^{\infty}|FKx(\varphi(t))|dt$$

$$\le\int_0^{\infty}\left[\exp(-t)+b\left|\int_0^{\infty}k(t-s)x(\varphi(s))ds\right|\right]dt$$

$$=\int_0^{\infty}\exp(-t)dt+b\|Kx(\varphi)\|$$

$$\le 1+b\|K\|\|x(\varphi)\|$$

$$=1+b\|K\|\int_0^{\infty}|x(\varphi(s))|ds$$

$$\le 1+b\|K\|\int_0^{\infty}|x(s)|ds$$

$$=1+b\|K\|\|x\|,$$

hence, for $x\in B_r$, we have

$$\|Gx\|\le 1+b\|K\|r$$

if we take $r=1+b\|K\|r$, then $r=\frac{1}{1-b\|K\|}$. This implies that G maps the ball B_r into itself, where

$$r=\frac{1}{1-b\|K\|}.$$

Step 3: For any $X\subset Q_r$ consider $x\in X$ and $\varepsilon>)$ be arbitrary, let be $D\subset R_+$ with $meas(D)\le\varepsilon$, then we have

$$\int_D|Gx(t)|dt=\int_D|FKx(\varphi(t))|dt$$

$$=\int_D\left[|bKx(\varphi(t))+exp(-t)|\right]dt$$

$$\le b\int_D|Kx(\varphi(t))|dt+\int_D\exp(-t)dt$$

$$\le b\int_D|Kx(\varphi(t))|dt-\exp(-t)meas(D)$$

$$\le b\int_D|Kx(\varphi(t))|dt-\exp(-t)\varepsilon$$

When ε tends to zero and from definition $C(X)$ that is a defined measure in the $L^1(R_+)$, we get $C(GX)\le bC(KX)$.

Step 4: For any $X\subset Q_r$ and $T>0$ we have

$$\int_T^{\infty}|Gx(t)|dt=\int_T^{\infty}|FKx(\varphi(t))|dt$$

$$=\int_T^{\infty}\left[|bKx(\varphi(t))+\exp(-t)|\right]dt$$

$$\le b\int_T^{\infty}|Kx(\varphi(t))|dt+\int_T^{\infty}\exp(-t)dt$$

$$\le b\int_T^{\infty}|Kx(\varphi(t))|dt+\exp(-T).$$

Now with take $\limsup_{T\to\infty}$ of the above inequality, we get

$$d(GX)\le bd(KX).$$

Where d is a defined measure in the $L^1(R_+)$. Now by step 3 and step 4 we deduce that.

$$\mu(GX)\le b\mu(KX).$$

Step 5: Take $x\in Q_r$ then $x(\varphi)$ is a.e. positive and nonincreasing on R_+ and consequently $Kx(\varphi)$ is also of the same type in virtue of the assumptions (i), (ii), (iii) and (iv) we deduce that $Gx=FKx(\varphi)$ is also a.e. positive and nonincreasing on R_+. This fact, together with the assertion $G:B_r\to B_r$ gives that G is a self-mapping of the set Q_r. For this reason that K is a linear and bounded operator, therefore K is continuous , and obviously F is a continuous operator then G is a continuous operator. Then K and G are continuous from Q_r into Q_r.

Step 6: We will show that $K(f(t,x))=f(t,K(x))$. We have $x(t)=Kx(t)$ if and only if $\exp(t)x(t)=\int_{t-1}^{t}\exp(s)x(s)ds$ therefore, if the function g satisfies $g'(t)=-\exp(-1)g(t-1)$,

then g is a fixed point of K. Hence, $g(t)=\exp(-t)$ is a fixed point of K. Thus

$$K(\exp(-t)+bx(t))=K(\exp(-t)+bK(x(t))$$

$$=\exp(-t)+bK(x(t))$$

$$=f(t,Kx(t)).$$

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Therefore, $K((Fx)(t)) = F((Kx)(t))$, i.e. K and F are commuting maps. For every $x \in Q_r$ we have $GK(x) = FKKx(\varphi) = KFKx(\varphi) = KG(x)$, so G and K are commuting maps Thus without the loss of generalities, in Corollary 2.4 enough that, we put $\varphi(t) = bt$ and $\mu(X) < \mu(KX)$. Then K and G have at least one common fixed point, which is a solution of the equation (7) and satisfies $x(t) = Kx(t)$. Moreover,

$$x(t) = \frac{\exp(-t)}{1-b}$$

is a common solution of the equations $f(t, x(t)) = x(t)$ and $K(x(t)) = x(t)$.

Conclusion

This paper examines the existence of a fixed point in various cases based on the measures of incompressibility, which is a very important technique in existence proof.

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